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SURFACE INSTABILITY IN GRADIENT ELASTICITY WITH SURFACE ENERGY

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Abstract—Biot's theory of plane strain surface instability of an isotropic elastic body under initial stress in finite strain is extended to include higher order strain-gradients. Higher order strain-gradients are properly introduced in the definition of the strain energy density, leading to an anisotropic gradient elasticity theory with surface energy. Accordingly, the present theory includes two material lengths characterizing the volume strain energy and the surface energy of the elastic body. The consideration of these two material lengths leads to the occurrence of a boundary layer. This in turn, gives rise to interesting phenomena related to the stability of the half-space, i.e. extra surface instability modes, thin skin effects and significant weakening of the half-space. It is also shown that the appearance of surface instability is associated with the vanishing velocity of propagation of Rayleigh waves. Furthermore, results derived in the context of the present theory on the dependence of the critical buckling stress of the layer on the thickness, suggest that it can be used effectively for the homogenization of elastic bodies containing periodic arrays of collinear Griffith cracks. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

It has been predicted theoretically that a homogeneously strained body with tractionless surfaces develops surface undulations or waves. This phenomenon is known as surface instability (Biot, 1965; Usmani and Beatty, 1974; Hill and Hutchinson, 1975; Hutchinson and Tvergaard, 1980; Vardoulakis, 1984). Since there are no physical length quantities in the continuum formulation of the problem, the wavelength of the surface instability mode can be arbitrarily short or long. The exponential decay beneath the surface is therefore also arbitrary, as it depends on the surface wavelength variation. In a more general formulation, conditions for the so-called "complementing condition" (Benallal *et al.*, 1988), for governing instabilities at the boundary of a solid have been established. In general, the complementing or consistency condition on boundary data (boundary conditions) in a boundary value problem, is a suitability condition of them to the governing differential equation (or to a system of governing differential equations).

Experiments on rock specimens reveal that failure under unconfined compression is usually manifested as exfoliation or slabbing due to surface instability. The fundamental fracture mechanism is the growth of small opening-mode splitting cracks oriented parallel to the free-surface. These cracks start at a shallow depth, and progress deeper into the rock with increased stress, leading to axial splitting. These cracks line up to form macroscopic splitting fractures and also form en echelon patterns that intersect the free surface. Eventually, slabs of a preferred slenderness fail by buckling (Fairhurst and Cook, 1966). Classical continuum mechanics theories cannot capture this phenomenon of surface degradation nucleating at a particular depth from the free surface because they do not have an intrinsic length scale. Furthermore, the continuum mechanics buckling loads predicted for the

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surface instability of a homogeneous, compressible half-space under horizontal compression are of the order of the shear modulus (Biot, 1965), and are therefore unrealistic; for example, the strength of many rock types is of the order of 1/100–1/1000 of their shear modulus. In order to overcome these inadequacies, Keer *et al.* (1982), Nazarenko (1986), and Vardoulakis and Papamichos (1991), have analyzed buckling under plane strain conditions due to horizontal compression of an elastic, half-space with pre-existing surface-parallel, co-planar cracks. They demonstrated that the critical buckling stress decreases dramatically as the distance between the free surface and the cracks diminishes. Furthermore, the influence of cracks far from the surface has little or no influence on the buckling stress. From these results it can be seen that a surface layer exists for which the presence of cracks influences the buckling stress significantly; this layer constitutes the non-homogeneous, bursting part of the material.

In terms of lattice theory, classical elasticity incorporates only the nearest neighbour interaction through the definition of the elastic strain energy density of the body, so it does not have an intrinsic length. An intrinsic length scale appears when the forces between particles are extended to include first, second, and nth neighbour interactions (Toupin and Gazis, 1964; Gazis and Wallis, 1965). Depending on the highest-order position gradient taken into account in the energy function, we call the corresponding material, an elastic material of grade-n. From this point of view the classical theory of elasticity may be considered as an asymptotic theory and the above ones as next order approximations. For the modelling of failure of solids by localization the significance of including a material length scale associated with the volume strain energy in continuum theories has been shown by Aifantis (1987) and Vardoulakis and Aifantis (1991). However, since localization in the form of cracks in brittle solids involves not only the volume strain energy but also the surface energy of newly created crack surfaces, the inclusion of surface energy in a continuum theory is of fundamental importance. An isotropic grade-3, linear elasticity theory with surface free energy was developed by Mindlin (1965). Mindlin's theory has been recently explored as far as its mathematical potential is concerned in a comprehensive paper by Wu (1992). Because this theory assumes material isotropy, it must include at least the first three gradients of the displacements (grade-3 theory), and consequently it must take into account double and triple forces per unit area before a material constant is introduced to capture the desired surface phenomena.

On the other hand, Casal (1961) was the first to see the connection between surface tension effects and the anisotropic grade-2 elasticity theory. Casal has extended the classical 1D Hookean definition of linear elastic solids by introducing additional terms of second order in displacements in the strain energy density expression (grade-2 theory) based on linear capillarity theory of liquids. Because this theory is anisotropic, it is possible to take into account surface free energy by multiplying a director (first-order tensor) with the strain-gradient (third-order tensor) in the expression for the strain energy density function. Accordingly, two material constants ℓ, ℓ' having the dimension of length, were introduced by Casal to characterize the internal and surface capillarity of the solid. The concept that the surfaces of liquids are in a state of tension is a familiar one, and it is widely utilized. The surface tension concept is therefore an analogy, but it explains the surface phenomena in solids in such satisfactory manner that the actual molecular phenomena need not be invoked. Hence, the present theory considers in an implicit manner molecular forces of cohesion acting upon the body. The additional components of the strain-gradient are accompanied by 27 components of double forces per unit area, which are stresses contributing neither resultant force nor couple, per unit area, across a surface in the material but, nevertheless, contributing to the potential energy and to the boundary conditions. It is worth noting that Mindlin's isotropic grade-3 theory includes 16 material constants plus the classical elasticity constants, whereas the present anisotropic grade-2 theory contains only two additional material constants, whose determination nonetheless constitutes a formidable experimental challenge. Also, a grade-2 theory is mathematically more tractable than a grade-3 theory.

Casal's original idea for the 1D tension bar problem has been generalized to three-dimensions by Vardoulakis and Sulem (1995) and applied to the solution of the mode-I, -

II and -III crack problems (Vardoulakis *et al.*, 1996; Exadaktylos *et al.*, 1996; Exadaktylos, 1997; Vardoulakis and Exadaktylos, 1997). It was found that the inclusion of the volume energy strain-gradient term ℓ in the constitutive equations predicts a crack shape forming cusps of the first kind with zero enclosed angle at the tip, which is consistent with Barenblatt's (1962) "cohesive-zone" theory but without requiring an extra assumption on the existence and effect of interatomic forces, as such effects are already incorporated in the stress–strain relation of the gradient elasticity. It was also shown that the effect of the volume strain-gradient term is to shield the applied loads leading to crack stiffening, whereas the effect of the surface energy strain-gradient term ℓ' is to amplify the applied loads leading to crack compliance by increasing the energy release rate of the crack.

In the present paper we focus on the influence of the volume and surface energy material lengths ℓ, ℓ', respectively, in plane strain surface instability of the isotropic half-space, of the single layer, and of a system of layers under pre-stress. Analysis of these types of instabilities incorporating only the volume energy strain-gradient term ℓ can be found in the papers by Frantziskonis and Vardoulakis (1992), Vardoulakis and Frantziskonis (1992), Benallal and Tvergaard (1995) and others. After a brief presentation of the equilibrium equations, boundary conditions, and stress and double stress constitutive equations, in Section 2, an exact incremental formulation in terms of the increment of the nominal or first Piola–Kirchhoff stress, referred and measured with respect to the initially stressed anisotropic gradient half-space with surface energy, is presented in Section 3. In Appendix A a solution is given for the simple case of the half-plane, characterized by only one kinematical degree-of-freedom, in the presence of initial stress, including formulae for the surface energy per unit area and for the strain—which decays exponentially into the interior. These results are applied in Appendix B to a semi-infinite strip (layer) of thickness h. The results for the single layer are used to derive a general formulation of the stability equations for multilayered media, whereas the numerical solution is achieved by using the transfer matrix technique. Finally, the discussion of numerical results is presented in Section 4.

2. THEORY

2.1. *Stress-equation of motion and boundary conditions*

Higher grade continua belong to a general class of constitutive models which account for the materials microstructure. An early formulation of a simple linear continuum theory with microstructure can be found in a rather unnoticed publication by Casal (1961), referred to by Germain (1973a,b). It is noted that Casal's model cannot be directly embedded in Mindlin's (1964) linear, isotropic elasticity theory with microstructure because the former is an anisotropic elasticity model. Instead, Casal's expression for the global strain energy of the one-dimensional tension bar was recovered by introducing an appropriate anisotropic, linear elastic, restricted Mindlin continuum. The theory is fully presented in Vardoulakis and Sulem (1995, Chapter 10), however, for easy reference we recapitulate here the basic equations.

Mindlin's theory (Mindlin, 1964) introduced the idea of the "unit cell" (micro-medium, which may be interpreted as the periodic structure of a crystal lattice, a molecule of a polymer, a crystal of a polycrystal, or a grain of a granular material. Appropriate kinematical quantities are then defined to describe geometrical changes in both the macro- and micro-medium. Next, the following Ansatz for the potential energy density w (potential energy per unit macro-volume) is taken

$$w = w(\varepsilon_{ij}, \gamma_{ij}, \kappa_{ijk}) \tag{1}$$

where $\gamma_{ij} \equiv \partial_i u_j - \psi_{ij}$ is the relative deformation (i.e. the difference between the macro-displacement-gradient and the micro-deformation), ψ_{ij} denotes the micro-deformation (i.e. the displacement-gradient in the micro-medium), $\kappa_{ijk} \equiv \partial_i \psi_{jk}$ denotes the micro-deformation gradient, and ε_{ij} is the usual strain (now the macro-strain) defined as follows

$$\varepsilon_{ij} \equiv \tfrac{1}{2}(\partial_j u_i + \partial_i u_j) \tag{2}$$

In eqn (2) u_i is the Cartesian component of the displacement vector and $\partial_k \equiv \partial/\partial x_k$, with x_k to denote space coordinates ($k = 1, 2, 3$). Then, appropriate definitions for the stresses follow from the variation of w

$$\tau_{ij} \equiv \frac{\partial w}{\partial \varepsilon_{ij}} = \tau_{ji}, \quad \alpha_{ij} \equiv \frac{\partial w}{\partial \gamma_{ij}}, \quad \mu_{ijk} = \frac{\partial w}{\partial \kappa_{ijk}} \tag{3}$$

where the second order stress tensor τ_{ij}, which is dual in energy to the macroscopic strain, is symmetric (i.e. $\tau_{ij} = \tau_{ji}$) and is called by Mindlin the Cauchy stress, the tensor α_{ij}, which is dual in energy to the relative deformation is asymmetric and is called the relative stress, and the third order tensor μ_{ijk}, which is dual in energy to the strain-gradient, is called the double stress.

We write Hamilton's principle for independent variations δu_i, $\delta\psi_{ij}$ between fixed limits of u_i and ψ_{ij} at times t_0 and t_1 (Love, 1927)

$$\delta \int_{t_0}^{t_1} (T - W)\,\mathrm{d}t + \int_{t_0}^{t_1} \delta W_1\,\mathrm{d}t = 0 \tag{4}$$

where T and W are the total kinetic and potential energies, respectively, and δW_1 is the variation of the work done by external forces. It can be shown that the following relationship is valid (Mindlin, 1964)

$$\delta \int_{t_0}^{t_1} T\,\mathrm{d}t = -\int_{t_0}^{t_1} \mathrm{d}t \int_V (\rho\, \partial_{tt} u_j \delta u_j + \tfrac{1}{3}\rho d^2\, \partial_{tt}\psi_{ij}\delta\psi_{ij})\,\mathrm{d}V \tag{5}$$

wherein ρ is the mass of macro-material per unit macro-volume, d is the half-edge length of the cube of the micro-medium, and ∂_t denotes differentiation with respect to time. Then, from the definition

$$\delta w = \tau_{ij}\delta\varepsilon_{ij} + \alpha_{ij}\delta\gamma_{ij} + \mu_{ijk}\delta\kappa_{ijk} \tag{6}$$

and by applying the divergence theorem

$$\delta W = \int_V \delta w\,\mathrm{d}V = -\int_V \partial_i(\tau_{ij} + \alpha_{ij})\delta u_j\,\mathrm{d}V - \int_V (\partial_i \mu_{ijk} + \alpha_{jk})\delta\psi_{jk}\,\mathrm{d}V + \int_{\partial V} n_i(\tau_{ij} + \alpha_{ij})\delta u_j\,\mathrm{d}S + \int_{\partial V} n_i \mu_{ijk}\delta\psi_{jk}\,\mathrm{d}S \tag{7}$$

where ∂V denotes the surface of the considered volume V, and n_k is the outward unit normal on the boundary ∂V. The structure of eqn (7) is the motivation for the adoption of the following form for the variation of the work done by external forces

$$\delta W_1 = \int_V f_j \delta u_j\,\mathrm{d}V + \int_V \Phi_{jk}\delta\psi_{jk}\,\mathrm{d}V + \int_{\partial V} (t_j \delta u_j + T_{jk}\delta\psi_{jk})\,\mathrm{d}S \tag{8}$$

where f_k is the body force per unit volume, t_k is the surface force per unit area (traction), Φ_{jk} is to be interpreted as a double force per unit volume, and T_{jk} as double force per unit area.

Substituting eqns (5), (7) and (8) in (4), and dropping the integration with respect to time, the variational equation of motion is obtained which subsequently yields the following 12 stress-equations of motion

$$\partial_i(\tau_{ij}+\alpha_{ij})+f_j = \rho\,\partial_{tt}u_j,$$
$$\bar{\alpha}_{jk}+\partial_i\mu_{ijk} = \tfrac{1}{3}\rho d^2\,\partial_{tt}\,\partial_j u_k, \quad \bar{\alpha}_{jk} \equiv \alpha_{jk}+\Phi_{jk} \tag{9}$$

and the 12 traction boundary conditions

$$n_j(\tau_{jk}+\alpha_{jk}) = t_k$$
$$n_i\mu_{ijk} = T_{jk} \tag{10}$$

Next, by defining the total stress tensor $\boldsymbol{\sigma}_{ij}$

$$\sigma_{ij} \equiv \tau_{ij}+\alpha_{ij} \tag{11}$$

we notice that according to eqn (9) the total stress tensor is identified with the common (macroscopic) equilibrium stress tensor. The τ_{ij} are like the components of the usual stress with the dimensions of force per unit area, however, they depend on the second gradient of strain in addition to the strain. The 27 components μ_{kij} have the character of double forces per unit area. The first subscript of a double stress μ_{kij} designates the normal to the surface across which the component acts; the second and third subscripts have the same significance as the two subscripts of σ_{ij} (Fig. 1). The eight components of the deviator of the couple-stress or couples per unit area formed by the combinations $1/2(\mu_{pqr}-\mu_{prq})$ are all equal to zero in the present gradient dependent elasticity theory, whereas all the remaining 10

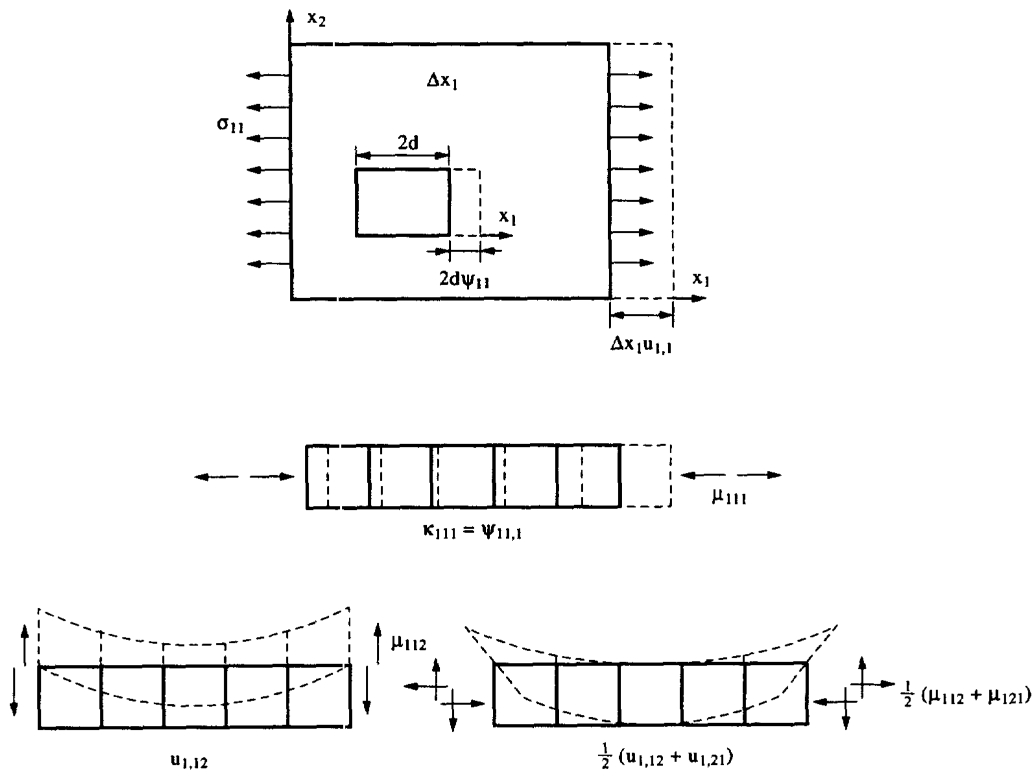

Fig. 1. Total stress σ_{11}, displacement gradient $u_{1,1} = \partial u_1/\partial x_1$, and double stresses μ_{111}, μ_{112} and $1/2(\mu_{112}+\mu_{121})$.

independent combinations $1/2(\mu_{pqr}+\mu_{prq})$ are self-equilibrating (Mindlin, 1964; 1965). Double force systems without moments are stress systems equivalent to two oppositely directed forces at the same point; such systems have direction but not net force and no resulting moment. Notice that singularities of this kind are discussed by Love (1927) and Eshelby (1951).

In case of a restricted Mindlin continuum, i.e. a micro-homogeneous material for which the macroscopic strain coincides with the micro-deformation, i.e.

$$\gamma_{ij} = \partial_i u_j - \psi_{ij} \equiv 0, \kappa_{ijk} \equiv \partial_i \varepsilon_{jk}$$

we cannot consider independent variations of δu_i alone since the $\partial_i \delta u_j$ are no longer independent of δu_i on ∂V because, if δu_i is known on ∂V, so is the surface-gradient of δu_i. Also, the relative stress tensor α_{ij} is workless, and the potential energy density function, w, takes the form

$$w = w(\varepsilon_{ij}, \partial_k \varepsilon_{ij}) \tag{12}$$

Furthermore, since we are dealing with single-valued displacement fields one can easily establish a one-to-one correspondence between $\partial_k \varepsilon_{ij}$ and $\partial_k \partial_j u_i$ (Mindlin and Eshel, 1968). The variation of the total potential energy in volume V of the body is defined as follows (Mindlin, 1964; 1965)

$$\delta \int_V w \, \mathrm{d}V = \int_V (\tau_{ij} \delta \varepsilon_{ij} + \mu_{ijk} \partial_i \delta \varepsilon_{jk}) \, \mathrm{d}V \tag{13}$$

where

$$\tau_{ij} = \frac{\partial w}{\partial \varepsilon_{ij}}, \quad \mu_{ijk} = \frac{\partial w}{\partial(\partial_i \varepsilon_{jk})} \tag{14}$$

To prepare for the formulation of a variational principle, we apply the chain rule of differentiation and the divergence theorem; furthermore, we resolve $\partial_i u_j$ on the boundary ∂V of V into a surface-gradient and a normal gradient

$$\begin{aligned} \partial_i \delta u_j &\equiv D_i \delta u_j + n_i D \delta u_j, \\ D_i &\equiv (\delta_{ik} - n_i n_k) \partial_k, \quad D \equiv n_k \partial_k \end{aligned} \tag{15}$$

where δ_{ij} is the Kronecker delta. The final expression for the variation in potential energy reads

$$\begin{aligned} \delta W = \int_V \delta w \, \mathrm{d}V = &-\int_V \partial_i(\tau_{ij} - \mu_{ijk,k}) \delta u_j \, \mathrm{d}V + \int_{\partial V} n_i(\tau_{ij} - \mu_{ijk,k}) \delta u_j \, \mathrm{d}S \\ &+ \int_{\partial V} L_i n_k \mu_{ijk} \delta u_j \, \mathrm{d}S + \int_{\partial V} n_i n_k \mu_{ijk} D \delta u_j \, \mathrm{d}S \end{aligned} \tag{16}$$

where $L_i = n_i D_k n_k - D_i$. Looking at the structure of eqn (16) we now postulate the following form for the variation of work W_1 done by external forces

$$\delta W_1 = \int_V f_j \delta u_j \, \mathrm{d}V + \int_{\partial V} (\tilde{P}_j \delta u_j + \tilde{R}_j D \delta u_j) \, \mathrm{d}S \tag{17}$$

where $\tilde{P}_k$, $\tilde{R}_k$ are the specified tractions and double tractions, respectively, on the smooth surface ∂V. From eqns (4), (5), (13), (14), (16) and (17) follow the stress-equations of motion in the volume V

$$\partial_i(\tau_{ij}-\partial_k\mu_{kij})+f_j = \rho\,\partial_{tt}u_j-\tfrac{1}{3}\rho d^2\,\partial_i(\partial_{tt}\,\partial_i u_j) \tag{18}$$

The surface ∂V of the considered volume V is divided into two complementary parts ∂V_u and ∂V_σ such that on ∂V_u kinematic data whereas ∂V_σ static data are prescribed. In classical continua these are constraints on displacements and tractions, respectively. For the stresses the following set of six traction boundary conditions on a smooth surface ∂V_σ is also derived from the virtual work principle

$$n_j\tau_{jk}-n_in_jD\mu_{ijk}-(n_iD_j+n_jD_i)\mu_{ijk}+(n_in_jD_\ell n_\ell-D_jn_i)\mu_{ijk}+n_in_j\mu_{ijk} = R_k \tag{19}$$

Since second-grade models introduce second strain gradients into the constitutive description, additional kinematic data must be prescribed on ∂V_u. With the displacement already given in ∂V_u, only its normal derivative with respect to that boundary is unrestricted. This means that on ∂V_u the normal derivative of the displacement should also be given, i.e.

$$u_i = w_i \quad \text{on} \quad \partial V_{u1} \quad \text{and} \quad Du_i = r_i \quad \text{on} \quad \partial V_{u2} \tag{20}$$

2.2. *Constitutive equations*

The most general form of the strain energy density function for a linear, anisotropic, grade-2 elastic material is

$$\begin{aligned} w &= \tfrac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj}+G\varepsilon_{ij}\varepsilon_{ji}+\ell_{1k}\,\partial_k(\varepsilon_{ii}\varepsilon_{jj})+\ell_{2k}\,\partial_k(\varepsilon_{ij}\varepsilon_{ji})\\ &+\ell_{3j}\,\partial_k(\varepsilon_{ij}\varepsilon_{ik})+\ell_{4k}\,\partial_i(\varepsilon_{ij}\varepsilon_{kj})+\ell_{5k}\,\partial_j(\varepsilon_{ii}\varepsilon_{kj})+c_1\,\partial_j\varepsilon_{ij}\,\partial_k\varepsilon_{ik}\\ &+c_2\,\partial_k\varepsilon_{ii}\,\partial_j\varepsilon_{kj}+c_3\,\partial_k\varepsilon_{ii}\,\partial_k\varepsilon_{jj}+c_4\,\partial_k\varepsilon_{ij}\,\partial_k\varepsilon_{ij}+c_5\,\partial_k\varepsilon_{ij}\,\partial_i\varepsilon_{kj}\end{aligned}$$

where λ and G are the usual Lamé constants, the five c_n are the additional constants which appear in Toupin's strain-gradient theory (Toupin, 1962; Mindlin, 1964), and ℓ_{nk} $(n = 1,\ldots,5, k = 1,\ldots,3)$ are the five additional constants resulting from the generalization of Casal's theory.

For the special form of gradient elasticity that we consider here we assume that ℓ_{2k} and c_4 are the only non-vanishing gradient coefficients. Thus, by setting $c_4 = G\ell^2$ and $\ell_{2k} = G\ell_k$ $(k = 1,\ldots,3)$, the three-dimensional generalisation of Casal's gradient-dependent anisotropic elasticity with surface energy leads to the following expression for the strain energy density function (Vardoulakis and Sulem, 1995)

$$w = \tfrac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj}+G\varepsilon_i\varepsilon_{ji}+G\ell^2\,\partial_k\varepsilon_{ij}\,\partial_k\varepsilon_{ji}+G\ell_k\,\partial_k(\varepsilon_{ij}\varepsilon_{ji}) \tag{21}$$

where ℓ,ℓ' are characteristic lengths of the material, and

$$\ell_k \equiv \ell' v_k, \quad v_kv_k = 1 \tag{22}$$

is a director. Accordingly eqn (21) defines a gradient anisotropic elasticity with constant characteristic directors ℓ_k. The last term in eqn (21) has the meaning of surface energy, since by using the divergence theorem

$$\int_V \partial_r(\ell_r\varepsilon_{pq}\varepsilon_{qp})\,\mathrm{d}V = \ell'\int_{\partial V}(\varepsilon_{pq}\varepsilon_{qp})(v_rn_r)\,\mathrm{d}S$$

If we require the energy density w, in eqn (21), to be positive definite (implying that energy is stored, rather than produced, as a result of deformation) for the simplest case of uniaxial deformation we find, as necessary and sufficient condition (see Appendix A)

$$-1 < \frac{\ell'}{\ell} < 1$$

Hence, the surface energy strain-gradient term ℓ' cannot exist alone, i.e. $\ell' \neq 0$ with $\ell = 0$, since in this case w becomes negative definite. On the other hand, the volume energy strain-gradient term can exist alone, i.e. $\ell \neq 0$ and $\ell' = 0$. It seems also reasonable to adopt $G > 0$, $3\lambda + 2G > 0$, as these are acceptable in the absence of double stresses.

From eqns (14), and (21), follow the constitutive relations for the Cauchy stress and double stress tensors, respectively

$$\tau_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2G(\varepsilon_{ij} + \ell_k\,\partial_k\varepsilon_{ij})$$
$$\mu_{kij} = 2G(\ell_k\varepsilon_{ij} + \ell^2\,\partial_k\varepsilon_{ij}) \tag{23}$$

Notice that the relative stress tensor α_{ij} can explicitly be obtained by recourse to the second and third of relations (9), and the second of eqn (23). Obtaining α_{ij}, permits, in turn, the determination of the total stress tensor σ_{ij} through eqn (23) and the definition equality (11), as follows

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2G(\varepsilon_{ij} - \ell^2\nabla^2\,\partial_k\varepsilon_{ij}) - \Phi_{ij} \tag{24}$$

3. SURFACE INSTABILITY

3.1. *Field equations and incremental constitutive equations*

In this paper, only plane strain surface instabilities will be discussed. It should be emphasized, however, that three-dimensional surface instabilities in the sense of Hutchinson and Tvergaard (1980) are also possible. These surface instabilities could yield eventually to lower critical stress but will not be discussed here. Consider the problem of the plane strain deformation of an isotropic semi-infinite body due to a large uniform compressive stress $\sigma_2 = -\sigma(\sigma > 0)$, parallel with the surface with outward unit normal vector $(\mathbf{n}_1\,\mathbf{n}_2) = (-1\;0)$, as shown in Fig. 2. Starting from a stress-free configuration, C_0, the body is stressed uniaxially under plane strain conditions, and C is the resultant configuration. In order to study the stability of continued equilibrium in C, the existence of a non-homogeneous infinitesimal transition, $C \rightarrow C'$, is investigated, with C always serving as the reference configuration. The displacement field for such a motion has the form

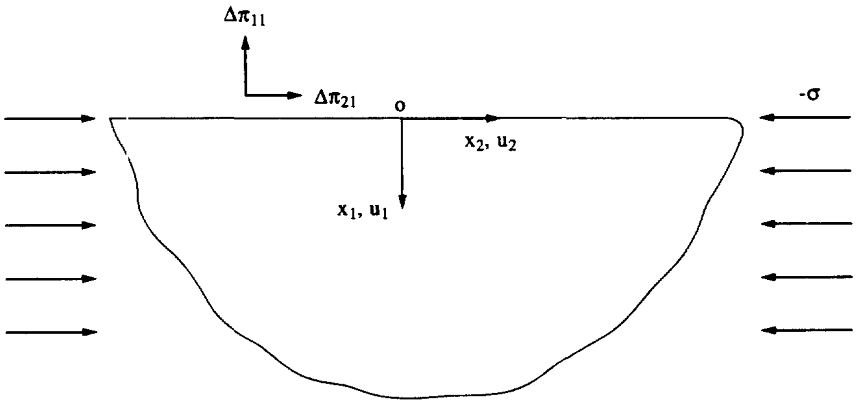

Fig. 2. Half-space under uniform compressive load and coordinates.

$$u_k = \hat{u}_k(x_1, x_2)e^{ft} \tag{25}$$

If an unbounded, non-periodic solution exists ($f > 0$), then the equilibrium in C, under dead loading conditions, is inherently unstable. This critical state with $f = 0$ marks the beginning of the regime of unstable solutions and is usually called the bifurcation state (Vardoulakis, 1984).

The problem under consideration is formulated in terms of the first Piola–Kirchhoff stress π_{ij} with respect to current configuration with $\Delta\pi_{ij}$ being its increment referred to the deformed initially stressed state C. In the case of constant body forces, the equations of motion for the incremental problem take the form (Beatty, 1966)

$$\partial_j \Delta\pi_{ij} = \rho\, \partial_{tt} u_i \tag{26}$$

Let $u_i(x_1, x_2)$, where x_i $(i = 1, 2)$ are the Cartesian coordinates, be the instantaneous incremental displacement components of a typical material point in the current configuration. The stress increment $\Delta\pi_{ij}$ is related to the Jaumann increments of the total stress $\overset{\circ}{\Delta}\sigma_{ij}$, the initial stress σ_{ij}, and the incremental strain and rotation (spin) as follows (Vardoulakis and Sulem, 1995)

$$\Delta\pi_{ij} = \overset{\circ}{\Delta}\sigma_{ij} + \omega_{ik}\sigma_{kj} - \sigma_{ik}\varepsilon_{kj} + \sigma_{kk} \tag{27}$$

where the incremental infinitesimal rotation tensor is defined as usually

$$\omega_{ij} \equiv \tfrac{1}{2}(\partial_j u_i - \partial_i u_j)$$

In plane strain conditions (i.e. $\partial_3 \equiv 0$) and in the coordinate system of principal axes of initial stress σ_{ij} in the plane of the deformation, the stress-equations of motion eqn (26) take the following form

$$\begin{aligned} \partial_1 \overset{\circ}{\Delta}\sigma_{11} + \partial_2 \overset{\circ}{\Delta}\sigma_{12} + (\sigma_1 - \sigma_2)\,\partial_2\omega_{21} &= \rho\,\partial_{tt}u_1 \\ \partial_1 \overset{\circ}{\Delta}\sigma_{21} + \partial_2 \overset{\circ}{\Delta}\sigma_{22} + (\sigma_1 - \sigma_2)\,\partial_1\omega_{21} &= \rho\,\partial_{tt}u_2 \end{aligned} \tag{28}$$

Assuming infinitesimal strain elasticity, the Jaumann stress increments $\overset{\circ}{\Delta}\sigma_{ij}$ of the total stress are related directly to the strain increments through the constitutive relations of linear elastic materials, perturbated properly according to eqn (24) in order to account for higher-order strain gradients

$$\begin{aligned} \overset{\circ}{\Delta}\sigma_{11} &= C_{11}\varepsilon_{11} + C_{12}\varepsilon_{22} - 2G_*\ell^2\nabla^2\varepsilon_{11} \\ \overset{\circ}{\Delta}\sigma_{22} &= C_{21}\varepsilon_{11} + C_{22}\varepsilon_{22} - 2G_*\ell^2\nabla^2\varepsilon_{22} \\ \overset{\circ}{\Delta}\sigma_{12} = \overset{\circ}{\Delta}\sigma_{21} &= 2G_*(\varepsilon_{12} - \ell^2\nabla^2\varepsilon_{12}) \end{aligned} \tag{29}$$

where C_{ij} $(i,j = 1, 2)$ and G_* are material moduli. It may be noted that for an isotropic medium $C_{11} - C_{12} = G_*$. Linear elasticity is justified for brittle rock-like materials since the failure strain for these materials is of the order of 10^{-3}–10^{-2} (Deere and Miller, 1966).

By using the constitutive relations eqn (29) the stress-equations of motion eqn (28) for $\sigma_1 = 0$ become

$$\begin{aligned} &c_{11}\,\partial_{11}u_1 + [g_* - \xi]\,\partial_{22}u_1 + [c_{12} + g_* + \xi]\,\partial_1\,\partial_2 u_2 \\ &-g_*\ell^2[2\,\partial_{1111}u_1 + 3\,\partial_{11}\,\partial_{22}u_1 + \partial_{222}\,\partial_1 u_2 + \partial_{2222}u_1 + \partial_{111}\,\partial_2 u_2] = (\rho/G)\,\partial_{tt}u_1 \\ &[c_{21} + g_* - \xi]\,\partial_1\,\partial_2 u_1 + [g_* + \xi]\,\partial_{11}u_2 + c_{22}\,\partial_{22}u_2 - g_*\ell^2[2\,\partial_{2222}u_2 + 3\,\partial_{11}\,\partial_{22}u_2 \\ &+ \partial_{111}\,\partial_2 u_1 + \partial_{1111}u_2 + \partial_{222}\,\partial_1 u_1] = (\rho/G)\,\partial_{tt}u_2 \end{aligned} \tag{30}$$

where we have introduced the following non-dimensional elastic constants and pre-existing lateral load, respectively

$$c_{ij} = C_{ij}/G, \quad (i = 1,2), \quad g_* = G_*/G$$
$$\xi = -\sigma_2/2G \tag{31}$$

The appearance of the volume energy strain-gradient parameter ℓ in the displacement-equations of motion eqn (30) indicates that the main difference between classical elasticity and gradient dependent elasticity lies in a boundary layer. This boundary layer along the bounding surface of the half-space gives rise to interesting phenomena as it will be shown later.

For the considered non-homogeneous deformation mode, the displacement field is assumed to be given in terms of two unknown amplitude functions of the dimensionless coordinate x (Vardoulakis and Sulem, 1995).

$$\hat{u}_1 = U(x)\cos(\beta y); \quad U(x) = Ae^{i\gamma x} \quad \hat{u}_2 = V(x)\sin(\beta y); \quad V(x) = Be^{i\gamma x}$$
$$i = \sqrt{-1}, \quad x = \frac{x_1}{L}, \quad y = \frac{x_2}{L} \quad \lim_{x\to\infty} U(x) = 0, \quad \lim_{x\to\infty} V(x) = 0 \tag{32}$$

where L is a reference length associated with the wavelength Λ of the deformation and β is a dimensionless wave number. The wavelength Λ of the deformation is inversely proportional to β such that $\Lambda = 2\pi L/\beta$. By substituting the displacement field eqn (32) in the partial differential equations eqn (30), we obtain two linear homogeneous algebraic equations with respect to the integration constants A, B

$$A[2g_*\hat{\beta}^2 Z^4 + (c_{11} + 3g_*\hat{\beta}^2)Z^2 + g_*\hat{\beta}^2 + g_* - \xi + \hat{f}^2]$$
$$- B[(c_{12} + g_* + \xi + g_*\hat{\beta}^2)iZ + g_*\hat{\beta}^2 iZ^3] = 0$$
$$A[g_*\hat{\beta}^2 iZ^3 + (c_{21} + g_* - \xi + g_*\hat{\beta}^2)iZ]$$
$$+ B[(g_* + \xi + 3g_*\hat{\beta}^2)Z^2 + g_*\hat{\beta}^2 Z^4 + c_{22} + 2g_*\hat{\beta}^2 + \hat{f}^2] = 0 \tag{33}$$

where we have set

$$Z = \frac{\gamma}{\beta}, \quad \hat{\beta} = \frac{\ell}{L}\beta, \quad \hat{f}^2 = \frac{\rho}{G}\left(\frac{f}{\beta}\right)^2 \tag{34}$$

From the above definitions it can be seen that as $\hat{\beta} \to 0$, then $\Lambda/\ell \to \infty$, or the wavelength cannot "see" the volume-energy material length ℓ. On the other hand, as $\hat{\beta} \to \infty$ the volume-energy material length is infinitely larger than the wavelength of the deformation. The case $\hat{\beta} = \pi$ corresponds to the shortest physically meaningful wavelength (i.e. $\Lambda \geqslant 2\ell$). For a non-trivial solution in terms of A and B, the determinant of the system of eqn (33) must vanish. This leads to the following characteristic equation

$$pZ^8 + qZ^6 + rZ^4 + sZ^2 + t = 0 \tag{35}$$

where

$$p = 2g^2{}_*\hat{\beta}^4, \quad q = g_*\hat{\beta}^2(8g_*\hat{\beta}^2 + c_{11} + 2\xi + 2g_*)$$
$$r = g_*\hat{\beta}^2(12g_*\hat{\beta}^2 - c_{12} - c_{21} + 2c_{22} + 3c_{11} + 2\xi + 2g_* + 3\hat{f}^2) + c_{11}(g_* + \xi)$$
$$s = g_*\hat{\beta}^2(8g_*\hat{\beta}^2 + 3c_{22} + 2c_{11} - c_{21} - c_{12} - 2\xi - 2g_* + 6\hat{f}^2) + c_{11}c_{22} - c_{12}c_{21}$$
$$- c_{12}(g_* - \xi) - c_{21}(g_* + \xi) + \hat{f}^2(1 + c_{11}) + \hat{f}^2 \tag{36}$$

$$t = g_*\hat{\beta}^2(2g_*\hat{\beta}^2 + c_{22} - 2\xi + 2g_* + 3\hat{f}^2) + c_{22}(g_* - \xi) + \hat{f}^4 - \hat{\omega}^2(1 + c_{22}) - \xi\hat{f}^2$$

At the bifurcation state ($\hat{f} = 0$) and for negligible gradient effects ($\ell \to 0$), eqn (35) reduces to the classical biquadratic equation for Z (Biot, 1965; Vardoulakis, 1984). The characteristic eqn (35) has eight roots $\pm Z_1$, $\pm Z_2$, $\pm Z_3$, $\pm Z_4$ which in turn correspond to eight solutions for the displacement field amplitudes $\hat{U}_i(x)$, $\hat{V}_i(x)$ $(i = 1, \ldots, 8)$. The complete solution for $U(x)$, $V(x)$ is then given as a linear combination of the basis functions

$$U(x) = \sum_{i=1}^{8} A_i \hat{U}_i(x), \quad V(x) = \sum_{i=1}^{8} K_i A_i \hat{V}_i(x) \tag{37}$$

where A_i $(i = 1, \ldots, 8)$ are integration constants to be found from the boundary conditions, and the constants K_i $(i = 1, \ldots, 8)$ are obtained from eqn (33) as follows

$$K_i = \frac{2g_*\hat{\beta}^2 Z_i^4 + (c_{11} - 3g_*\hat{\beta}^2)Z_i^2 + (g_* + g_*\hat{\beta}^2 - \xi + \hat{f}^2)}{g_*\hat{\beta}^2 i Z_i^3 + (c_{12} + g_* + g_*\hat{\beta}^2 + \xi) i Z_i}, \quad K_i \in C (i = 1, \ldots, 8) \tag{38}$$

3.2. *Classification of regimes and solutions of the field equations*

The character of the roots of eqn (35) depends on whether the stress and material parameters, and $\hat{\beta}$, or alternatively the values of the real coefficients p, q, r, s and t are given by eqn (36) at the instant of bifurcation ($\hat{f} = 0$), are such that the current state is in the elliptic, hyperbolic or parabolic regimes.

For the isotropic half-space the following relations for the elastic constants are valid

$$C_{11} = C_{22} = (\lambda + 2G), \quad C_{12} = C_{21} = \lambda, \quad G_* = G \tag{39}$$

Whereas according to eqn (31) the following normalised elastic moduli are obtained

$$c_{11} = c_{22} = \frac{2(1-\nu)}{1-2\nu}, \quad c_{12} = c_{21} = \frac{2\nu}{1-2\nu}, \quad g_* = 1 \tag{40}$$

by virtue of eqn (40) and the transformation

$$z \equiv Z^2 \tag{41}$$

the characteristic eqn (35) for zero value of the dynamic parameter $\hat{f}$ reduces to

$$\begin{aligned} &\hat{\beta}^4 z^4 + \hat{\beta}^2\left(4\hat{\beta}^2 + \frac{1-\nu}{1-2\nu} + 1 + \xi\right)z^3 + \left[\hat{\beta}^2\left(6\hat{\beta}^2 + \frac{5-7\nu}{1-2\nu} + 1 + \xi\right)\right. \\ &\left. + \frac{(1-\nu)(1+\xi)}{1-2\nu}\right]z^2 + \left[\hat{\beta}^2\left(4\hat{\beta}^2 + \frac{5-7\nu}{1-2\nu} + 1 - \xi\right) + \frac{2(1-\nu)}{1-2\nu}\right]z \\ &+ 2\hat{\beta}^2\left(\hat{\beta}^2 + \frac{1-\nu}{1-2\nu} + 1 - \xi\right) + \frac{(1-\nu)(1-\xi)}{1-2\nu} = 0 \end{aligned} \tag{42}$$

The above equation has the following roots

$$z_1 = -1, z_2 = -\frac{1}{2}\frac{2\hat{\beta}^2 + 1 + \xi - \sqrt{8\hat{\beta}^2\xi + (1+\xi)^2}}{\hat{\beta}^2}$$

$$z_3 = -\frac{1}{2}\frac{2\hat{\beta}^2 + 1 + \xi + \sqrt{8\hat{\beta}^2\xi + (1+\xi)^2}}{\hat{\beta}^2}, \quad z_4 = -\left[1 + \frac{1-\nu}{(1-2\nu)\hat{\beta}^2}\right] \tag{43}$$

The root $z_1^c = -1$ appears also in the classical theory, whereas in the same theory which is characterized by only two roots, the second root is the $z_2^c = -(1-\xi)/(1+\xi)$. Of considerable importance in this problem is the behaviour of the roots eqn (43). The roots z_1, z_3, z_4 possess always negative real values, whereas the z_2-root has a negative or positive sign depending on the values of $\hat{\beta}, \xi$. According to the above, the possibilities for solving eqn (35) are classified as follows:

(EI) elliptic-imaginary subregime: All the roots z_i $(i = 1, \ldots, 4)$ are negative real if the following inequality holds true

$$\xi < 1+\hat{\beta}^2 \tag{44}$$

(note that in the absence of strain-gradients the necessary condition that all the roots z_i^c $(i = 1, 2)$ are negative is $\xi < 1$). Thus, by virtue to the transformation eqn (41) the characteristic eqn (35) has eight pure imaginary solutions $\pm\sqrt{z_i}$ $(i = 1, \ldots, 4)$.

(P) Parabolic regime: if eqn (44) is not valid then the roots z_1, z_3 and z_4 are negative real and z_2 is positive real, whereas according to the transformation eqn (41) the characteristic eqn (35) has six pure imaginary roots $\pm\sqrt{z_1}$, $\pm\sqrt{z_3}$, $\pm\sqrt{z_4}$ and two real roots $\pm\sqrt{z_2}$.

Figure 3 illustrates the characteristic regimes of the characteristic eqn (35) for the special case of the isotropic half-space considered here and for zero value of the dynamic parameter $\hat{f}$. Then, the solutions in the various regimes can be found to be:

(P) Parabolic regime: the notion of surface instability means that the deformation is confined close to the surface; i.e. the displacement field is fading exponentially with x, being zero at infinite x, thus

$$\lim_{x\to\infty} \hat{U} = 0, \quad \lim_{x\to\infty} \hat{V} = 0 \quad \text{or} \quad \mathrm{Im}\{Z_j\} > 0, \quad j = 1, \ldots, 8 \tag{45}$$

Since solution (32) for real γ cannot satisfy the boundness condition eqn (45), surface instabilities in (P) are not possible, however, solutions in the parabolic regime are those associated with internal buckling of a confined medium (Biot, 1963; 1965). Surface instabilities are only possible in the elliptic regime. For the special case of $\xi = 1+\hat{\beta}^2$ the roots $\pm\sqrt{z_2}$ are equal to zero, and by setting $A_2 = 0$ the boundness condition eqn (45) is satisfied.

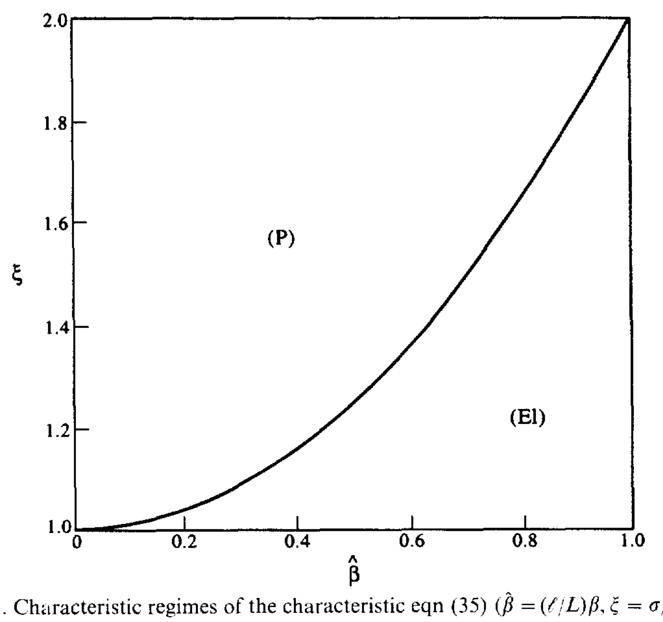

Fig. 3. Characteristic regimes of the characteristic eqn (35) ($\hat{\beta} = (\ell/L)\beta$, $\xi = \sigma/2G$).

However, in this case we cannot satisfy the four boundary conditions on the free surface since we have only three integration constants A_i $(i = 1, 3, 4)$.

From the classification of regimes given above, the solution in the (EI) subregime is as follows:

(EI)—subregime: Let $\hat{Z}_i = \mathrm{Im}(\sqrt{z_i})$ $(i = 1, \ldots, 4)$ where Im(·) denotes the imaginary value of what is enclosed in the parenthesis. Since $z_i < 0$ $(i = 1, \ldots, 4)$ then the exponentially fading solutions are of the form

$$
\begin{aligned}
U(x) &= A_1 e^{-\beta \hat{Z}_1 x} + A_2 e^{-\beta \hat{Z}_2 x} + A_3 e^{-\beta \hat{Z}_3 x} + A_4 e^{-\beta \hat{Z}_4 x} \\
V(x) &= K_1 A_1 e^{-\beta \hat{Z}_1 x} + K_2 A_2 e^{-\beta \hat{Z}_2 x} + K_3 A_3 e^{-\beta \hat{Z}_3 x} \\
&\quad + K_4 A_4 e^{-\beta \hat{Z}_4 x}, \quad \hat{Z}_i \quad (i = 1, \ldots, 4) \in \Re^+
\end{aligned} \tag{46}
$$

with

$$
\begin{aligned}
\hat{Z}_1 &= 1 \\
\hat{Z}_2 &= \frac{\sqrt{2}}{2\hat{\beta}} \sqrt{2\hat{\beta}^2 + 1 + \xi - \sqrt{8\hat{\beta}^2 \xi + (1+\xi)^2}} \\
\hat{Z}_3 &= \frac{\sqrt{2}}{2\hat{\beta}} \sqrt{2\hat{\beta}^2 + 1 + \xi + \sqrt{8\hat{\beta}^2 \xi + (1+\xi)^2}} \\
\hat{Z}_4 &= \sqrt{1 + \frac{1-v}{(1-2v)\hat{\beta}^2}}.
\end{aligned} \tag{47}
$$

The constant coefficients K_i $(i = 1, \ldots, 4)$ appearing in eqn (46) can be found by virtue of eqns (38), (40), and (47) as follows

$$
K_1 = \frac{\dfrac{2(1-v)}{1-2v} - 1 + \xi}{\dfrac{2v}{1-2v} + 1 + \xi}
$$

$$
K_2 = \frac{\dfrac{1}{8}\dfrac{a_1^2}{\hat{\beta}^2} - \dfrac{1}{4}\dfrac{\left(\dfrac{2(1-v)}{1-2v} + 3\hat{\beta}^2\right) a_1}{\hat{\beta}^2} + \hat{\beta}^2 + 1 - \xi}{\dfrac{1}{8}\dfrac{a_1^{3/2}}{\hat{\beta}} - \dfrac{1}{2}\dfrac{\left(\dfrac{2v}{1-2v} + \hat{\beta}^2 + 1 + \xi\right)\sqrt{a_1}}{\hat{\beta}}}
$$

$$
K_3 = \frac{\dfrac{1}{8}\dfrac{a_2^2}{\hat{\beta}^2} - \dfrac{1}{4}\dfrac{\left(\dfrac{2(1-v)}{1-2v} + 3\hat{\beta}^2\right) a_2}{\hat{\beta}^2} + \hat{\beta}^2 + 1 - \xi}{\dfrac{1}{8}\dfrac{a_2^{3/2}}{\hat{\beta}} - \dfrac{1}{2}\dfrac{\left(\dfrac{2v}{1-2v} + \hat{\beta}^2 + 1 + \xi\right)\sqrt{a_2}}{\hat{\beta}}}
$$

$$K_4 = \frac{-2\dfrac{[1-v+\hat{\beta}^2(1-2v)]^2}{\hat{\beta}^2(1-2v)^2} + \dfrac{\left(\dfrac{2(1-v)}{1-2v}+3\hat{\beta}^2\right)a_3}{(-1+2v)\hat{\beta}^2} + \hat{\beta}^2+1-\xi}{-\dfrac{[(1-2v)a_3]^{3/2}}{\hat{\beta}^2(1-2v)^3} + \dfrac{\left(\dfrac{2v}{1-2v}+\hat{\beta}^2+1+\xi\right)\sqrt{a_4}}{(1-2v)\hat{\beta}}} \tag{48}$$

with

$$\begin{aligned} a_1 &= -2(1+\xi)-4\hat{\beta}^2+2\sqrt{(1+\xi)^2+8\hat{\beta}^2\xi}, \\ a_2 &= -2(1+\xi)-4\hat{\beta}^2-2\sqrt{(1+\xi)^2+8\hat{\beta}^2\xi}, \\ a_3 &= -(1-v)-\hat{\beta}^2+2\hat{\beta}^2 v \\ a_4 &= (1-2v)a_3 \end{aligned} \tag{49}$$

3.3. *Boundary conditions*

For the plane problems we wish to consider here, the stress increments $\Delta\pi_{ij}$ can be found from eqns (27) and (29) (for $\sigma_1 = 0$) as follows

$$\begin{aligned} \Delta\pi_{11} &= C_{11}\,\partial_1 u_1 + C_{12}\,\partial_2 u_2 - 2G_*\ell^2(\partial_{111}u_1 + \partial_{22}\,\partial_1 u_1) \\ \Delta\pi_{22} &= (C_{21}+\sigma_2)\,\partial_1 u_1 + C_{22}\,\partial_2 u_2 - 2G_*\ell^2(\partial_{11}\partial_2 u_2 + \partial_{222}u_2) \\ \Delta\pi_{12} &= \left(G_* + \frac{\sigma_2}{2}\right)\partial_2 u_1 + \left(G_* + \frac{\sigma_2}{2}\right)\partial_1 u_2 \\ &\quad - G_*\ell^2(\partial_{11})\,\partial_2 u_1 + \partial_{222}u_1 + \partial_{111}u_2 + \partial_{22}\,\partial_1 u_2) \\ \Delta\pi_{21} &= \left(G_* - \frac{\sigma_2}{2}\right)\partial_2 u_1 + \left(G_* + \frac{\sigma_2}{2}\right)\partial_1 u_2 \\ &\quad - G_*\ell^2(\partial_{11}\,\partial_2 u_1 + \partial_{222}u_1 + \partial_{111}u_2 + \partial_{22}\,\partial_1 u_2) \end{aligned} \tag{50}$$

It is worth noting that the above stress-displacement relations are anisotropic and non-symmetric, i.e. $\Delta\pi_{12} \neq \Delta\pi_{21}$.

The relevant incremental traction boundary conditions for the half-space problem, i.e. $\Delta\pi_{ij}n_{ij} = 0$ $(i,j = 1,2)$, along the free-surface $(\mathbf{n}_1\,\mathbf{n}_2) = (-1\;0)$ take the form

$$\Delta\pi_{11} = 0, \quad \Delta\pi_{21} = 0, \quad -\infty < x_2 < \infty, \quad x_1 = 0 \tag{51}$$

whereas the double stress boundary conditions are specified as follows

$$\Delta\mu_{111} = 0, \quad \Delta\mu_{112} = 0, \quad -\infty < x_2 < \infty, \quad x_1 = 0 \tag{52}$$

For the surface instability problem the only boundary condition on incremental displacements or stresses at infinity is that given by eqn (45); i.e. we do not need to specify boundness of the stress increments $|x_2| \to \infty$. Then, by virtue of the second of constitutive relations eqn (23) the double stresses can be found as follows

$$\begin{aligned} \mu_{111} &= 2G_*(\ell'\varepsilon_{11} + \ell^2\,\partial_1\varepsilon_{11}) \\ \mu_{112} &= 2G_*(\ell'\varepsilon_{12} + \ell^2\,\partial_1\varepsilon_{12}) \end{aligned} \tag{53}$$

where we have set $v_1 \equiv -n_1$ in the definition equality eqn (22).

In the (EI)—subregime the boundary conditions eqns (51), (52) take the form, respectively

$$A_1\{2\hat{\beta}^2\hat{Z}_1(\hat{Z}_1^2-1)+(K_1c_{12}-\hat{Z}_1c_{11})\}+A_2\{2\hat{\beta}^2\hat{Z}_2(\hat{Z}_2^2-1)+(K_2c_{12}-\hat{Z}_2c_{11})\}$$
$$+A_3\{2\hat{\beta}^2\hat{Z}_3(\hat{Z}_3^2-1)+(K_3c_{12}-\hat{Z}_3c_{11})\}$$
$$+A_4\{2\hat{\beta}^2\hat{Z}_4(\hat{Z}_4^2-1)+(K_4c_{12}-\hat{Z}_4c_{11})\}=0 \tag{54}$$

$$A_1\{(K_1\hat{Z}_1+1)[\hat{\beta}^2(1-\hat{Z}_1^2)+(1+\xi)\}+A_2\{(K_2\hat{Z}_2+1)[\hat{\beta}^2(1-\hat{Z}_2^2)+(1+\xi)\}$$
$$+A_3\{(K_3\hat{Z}_3+1)[\hat{\beta}^2(1-\hat{Z}_3^2)+(1+\xi)\}+A_4\{(K_4\hat{Z}_4+1)[\hat{\beta}^2(1-\hat{Z}_4^2)+(1+\xi)\}=0 \tag{55}$$

$$A_1\{\hat{\beta}^2\hat{Z}_1^2-\hat{\beta}'\hat{Z}_1\}+A_2\{\hat{\beta}^2Z_2^2-\hat{\beta}'\hat{Z}_2\}+A_3\{\hat{\beta}^2\hat{Z}_3^2-\hat{\beta}'\hat{Z}_3\}+A_4\{\hat{\beta}^2Z_4^2-\hat{\beta}'\hat{Z}_4\}=0 \tag{56}$$

$$A_1\{\hat{\beta}^2\hat{Z}_1(K_1\hat{Z}_1+1)\hat{Z}_1-\hat{\beta}'(1+K_1\hat{Z}_1)\}+A_2\{\hat{\beta}^2\hat{Z}_2(K_2\hat{Z}_2+1)\hat{Z}_2-\hat{\beta}'(1+K_2\hat{Z}_2)\}$$
$$+A_3\{\hat{\beta}^2\hat{Z}_3(K_3\hat{Z}_3+1)\hat{Z}_3-\hat{\beta}'(1+K_3\hat{Z}_3)\}+A_4\{\hat{\beta}^2\hat{Z}_4(K_4\hat{Z}_4+1)\hat{Z}_4-\hat{\beta}'(1+K_4\hat{Z}_4)\}=0 \tag{57}$$

where we have introduced the surface energy strain-gradient parameter

$$\hat{\beta}'=\frac{\ell'}{L}\beta \tag{58}$$

Equations (54)–(57) form a homogeneous system of equations in terms of the constants A_1, A_2, A_3 and A_4, which can be put in the compact form

$$[Y]\{A\}=0 \tag{59}$$

To arrive at non-trivial solutions of the homogeneous system eqn (59), the matrix [**Y**] must be singular, that is either the determinant of the system must be zero, or one of its four eigenvalues must vanish, i.e.

$$J(\xi;\hat{\beta},\hat{\beta},v)=\det([Y])=0 \quad \text{or } \lambda_1\lambda_2\lambda_3\lambda_4=0 \tag{60}$$

where λ_i $(i=1,\ldots,4)$ are the four eigenvalues of the 4×4 matrix $[Y]$.

Equations (35), (38) and (54)–(57) or (60) constitute the formal solution of the problem. In eqn (60) we fix the wave numbers $\hat{\beta},\hat{\beta}'$, as well as Poisson's ratio v, and we consider it as an equation for the dimensionless load ξ. Accordingly, if $J(\xi)$ are values of the determinant that depend on ξ and on $\hat{\beta},\hat{\beta}',v$ as well, we write $J(\xi;\hat{\beta},\hat{\beta}',v)$ in place of $J(\xi)$. When monotonic loading is assumed, then the lowest level associated with a change of sign of the determinant or of one of the eigenvalues provides the critical buckling load.

Herein, the following eigenstrains are recorded

$$\hat{\varepsilon}_{11}(\hat{x},\hat{y})=-\sum_{i=1}^{4}A_i\hat{Z}_ie^{-\hat{\beta}\hat{Z}_i\hat{x}}\cos\hat{\beta}\hat{y}$$
$$\hat{\varepsilon}_{22}(\hat{x},\hat{y})=-\sum_{i=1}^{4}K_iA_ie^{-\hat{\beta}\hat{Z}_i\hat{x}}\cos\hat{\beta}\hat{y} \tag{61}$$

whereas the eigenstrain energy density is given by

$$\hat{w}(\hat{x},\hat{y}) = \sum_{i=1}^{4} A_i^2 \Bigg\{ [(K_i^2+\hat{Z}_i^2)((m+1)+\hat{\beta}^2\hat{Z}_i^2-2\hat{\beta}'\hat{Z}_i)$$

$$+2mK_iZ_i+\frac{\hat{\beta}^2}{2}(1+K_i\hat{Z}_i)^2\cos^2\hat{\beta}\hat{y}$$

$$+[\tfrac{1}{2}(1+K_i\hat{Z}_i)^2(1+\hat{\beta}^2\hat{Z}_i^2+2\hat{\beta}'\hat{Z}_i)+\hat{\beta}^2(K_i^2+\hat{Z}_i^2)]\sin^2\hat{\beta}\hat{y}\Bigg\}\exp(-2\hat{\beta}\hat{Z}_i\hat{x}) \quad (62)$$

where we have put

$$\hat{\varepsilon}_{ij} = \varepsilon_{ij}/(\hat{\beta}/\ell), \quad \hat{x} = x_1/\ell, \quad \hat{y} = x_2/\ell, \quad m = \frac{v}{1-2v} \quad (63)$$

Finally, the analysis of an isolated layer and of multilayered systems in the context of the present gradient dependent elasticity theory with initial stress is presented in Appendix B.

4. DISCUSSION OF THE SURFACE INSTABILITY FOR THE GRADIENT-ELASTIC BODY WITH SURFACE ENERGY

The numerical results that refer to the buckling of the homogeneous, isotropic, compressible half-space with $\ell \to 0$ under horizontal compression are presented in Fig. 4. In this figure the dimensionless buckling load ξ defined by eqn (31) is plotted against the Poisson's ratio v of the half-space. The numerical results presented also in Table 1 agree exactly with the analytical solution (Biot, 1965) to the problem.

By considering the equations of motion of the gradient-dependent elastic half-space under initial stress and by seeking these solutions that are confined close to the free surface, one can find the critical stresses ξ that characterize dynamic or quasi-static surface buckling, respectively. Such a computation is shown in Fig. 5, where the dimensionless pre-stress ξ is plotted vs the dynamic coefficient $\hat{f}^2$, defined by eqn (34), for $\hat{\beta} = 0.2$, $\hat{\beta}' = 0$, $v = 0.25$. As it is shown in the same figure, there exist two distinct solutions belonging to the branches I and II, namely $\xi_1 = 0.4747$ and $\xi_2 = 0.7081$, respectively, that correspond to the bifurcation state since they mark the beginning of the regime of unstable solutions ($f > 0$). On the other hand, harmonic solutions in the pre-bifurcation regime (f imaginary) correspond to

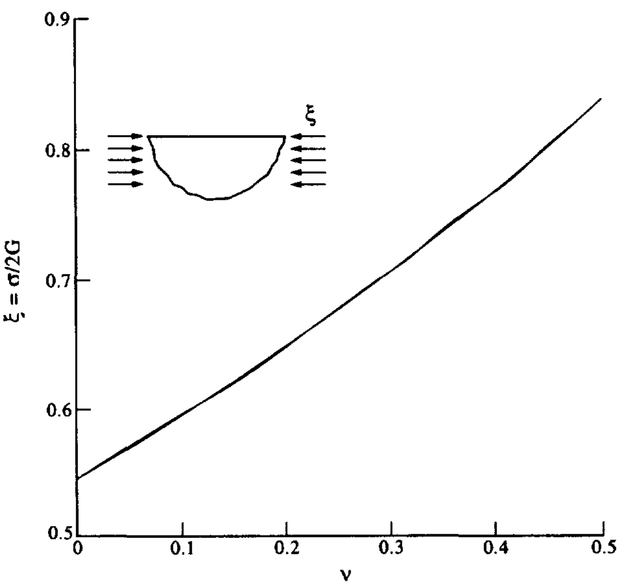

Fig. 4. Dimensionless buckling load of the elastic half-space as a function of the Poisson's ratio v for $\beta \to 0$ and zero value of the surface energy parameter (i.e. $\hat{\beta}' = 0$).

Table 1. Dimensionless buckling load $\xi = \sigma/2G$ for various values of the Poisson's ratio $\nu(\beta \to 0, \hat{\beta}' = 0)$

Poisson's ratio (ν)	Dimensionless buckling load ($\xi = \sigma/2G$)
0	0.544
0.05	0.568
0.1	0.594
0.15	0.621
0.2	0.650
0.25	0.680
0.3	0.712
0.35	0.744
0.4	0.776
0.45	0.808
0.499	0.839

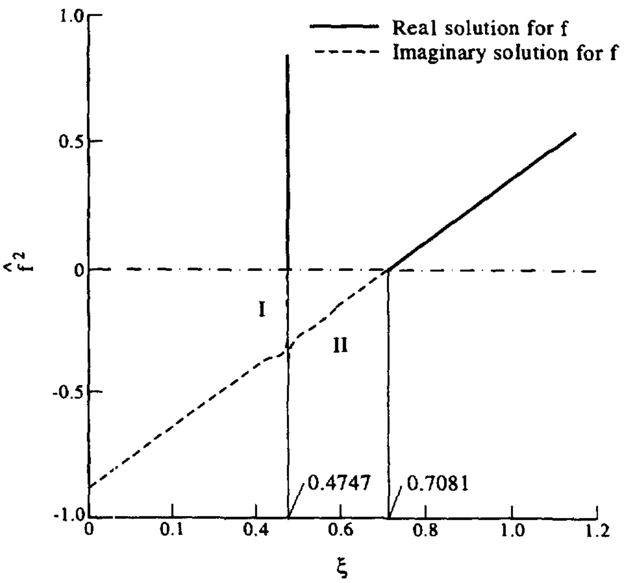

Fig. 5. Dynamic coefficient $\hat{f}^2$ as a function of the dimensionless lateral compression ξ for $\nu = 0.25$, $\hat{\beta} = 0.2$, $\hat{\beta}' = 0$.

the propagation of Rayleigh surface waves under pre-stress [see also Iliadis (1996)]. Figure 5 demonstrates that as the compressive pre-stress ξ increases, the velocity of propagation of Rayleigh surface waves [the ratio of Rayleigh wave to shear wave velocity is proportional to $\text{Im}(\hat{f})$] decreases, following the two branches I and II, and finally reaches the zero value that corresponds to the mode-1 and mode-2 buckling loads of the half-space. This is in accordance with Biot (1963) and with the "complementing condition" (Benallal *et al.*, 1988) that is, the appearance of surface buckling which is associated with the vanishing of the velocity of propagation of surface waves, marks the onset of ill-posedness of the underlying generic initial boundary value problem of elasticity under initial stress. It is worth noting that there are two surface waves: one propagating (with real wave number) that corresponds to the imaginary part of branch II, the other non propagating or standing wave (with imaginary wave number) that corresponds to the imaginary part of branch I. This is demonstrated by the infinite slope of branch I at $\xi \cong 0.47$. The evolution of the two distinct branches I, II is a major source of the difference between elastic fields with and without double stresses. In both dynamic and static fields, it contributes boundary layer effects and singularities.

In Fig. 6a the value of the determinant of the square matrix $[Y]$ in eqn (59) is plotted against the dimensionless pre-stress ξ for $\hat{\beta} = 0.4$, $\hat{\beta}' = 0$, $v = 0.25$. Also, in Fig. 6b and 6c the real and imaginary values of the four eigenvalues of matrix $[Y]$ are also plotted against ξ. From these figures it is clear that the isotropic half-space that is characterized by volume energy strain-gradient only, exhibits two non-trivial buckling modes at $\xi_1 = 0.412$ and $\xi_2 = 0.787$, respectively. On the other hand, the classical elastic half-space under initial stress is characterized by only one non-trivial buckling mode (Biot, 1963). Furthermore, the dependence of the dimensionless buckling load ξ on the volume energy strain-gradient parameter $\hat{\beta}$ for $\hat{\beta}' = 0$, $v = 0.25$ corresponding to the two distinct buckling modes is shown in Fig. 7. In the same figure the classical solution—which predicts that buckling load is independent of wave number of scale—is also plotted with a dashed line. It can be seen that for the first buckling mode the buckling stress ξ decreases, whereas for the second mode it decreases for increasing values of $\hat{\beta}$. The existence of the extra first buckling mode even in the limit as $\hat{\beta} \to 0$ is attributed to the existence of a boundary layer along the free surface of the half-space since the extra boundary data cannot be satisfied by the limit classical equation [see for example Cole (1968)].

The distributions of the eigenstrains $\hat{\varepsilon}_{11}, \hat{\varepsilon}_{22}$ beneath the free surface corresponding to the first and second buckling modes are shown in Fig. 8a and 8b, respectively. Also, in the same figures the distribution of the normalized eigenstrain energy density w is presented graphically with a dashed line. From Fig. 8a it is observed that the first buckling mode is manifested by a strain localization at a thin skin near the free surface that decays exponentially into the interior. Hence, the basic feature of the present gradient dependent elasticity theory is the appearance of a skin effect (see Appendix A). On the other hand, the distribution of $\hat{\varepsilon}_{11}$ corresponding to the second mode (Fig. 8b) is characterized by a lower amplitude and attenuation rate compared to those of the first mode, whereas it predicts that the onset of the buckling will start from a certain point beneath the surface and at a depth which is related to the characteristic material length ℓ. This means that higher buckling modes penetrate deeper than do lower modes. Furthermore, the comparison of the normal on the bounding surface eigenstrain $\hat{\varepsilon}_{11}$ associated with the first (1st) and second (2nd) buckling mode in Fig. 8a and 8b, respectively, shows that a significant attenuation of $\hat{\varepsilon}_{11}$ in the first mode at a depth of approximately 5ℓ marks the maximum value of $\hat{\varepsilon}_{11}$ in the second mode. The previous results show that the present gradient dependent theory essentially captures the fundamental fracture mechanism of brittle rock specimens or structures (free surfaces of underground openings in rock, rock pillars etc.) in uniaxial compression (Fairhurst and Cook, 1966), that is the growth of opening-mode splitting cracks oriented parallel to the free surface, starting very close to it (1st buckling mode) and progressing deeper into the rock with increased stress (2nd buckling mode). Figure 9 shows the effect of the volumetric strain-gradient wave number $\hat{\beta}$ on the distribution of the eigenstrain $\hat{\varepsilon}_{11}$ associated with the 2nd buckling mode for a zero value of the surface energy parameter $\hat{\beta}'$. From the same figure it can be observed that as $\hat{\beta}$ increases, the maximum eigenstrain decreases, or the material becomes stiffer.

From uniaxial compressive loading of cylindrical or prismatic rock specimens or from plane strain tests by using the Surface Instability Detection apparatus (Papamichos *et al.*, 1994) the depth of spalling or surface degradation can be obtained which then provides ℓ. Such a calibration of the volumetric strain-gradient parameter ℓ is of paramount importance if one wishes to solve surface instability problems of either single-layer or more geologically meaningful multilayered specimens by using constitutive equations of the present type.

Next, in order to investigate the influence of the surface energy term ℓ' in the surface instability, the real values of the eigenvalues associated with the surface buckling modes for the specific case of $\hat{\beta} = 0.4$, $v = 0.25$, are plotted in Fig. 10 against the dimensionless lateral stress ξ with continuous and dashed lines for $\ell'/\ell = 0$, $\ell'/\ell = 1.1$, respectively. In the latter case the presence of the surface energy ($\ell'/\ell = 1.1$) results in a noticeable lower value of the buckling stress compared to that predicted by the local continuum theory. It can be seen that in addition to the two buckling modes at $\xi = 0.412$ and $\xi = 0.787$ that have been found for zero value of the surface energy parameter, another eigenvalue changes sign at a lower load $\xi = 0.055$, which is also the critical buckling load since monotonic

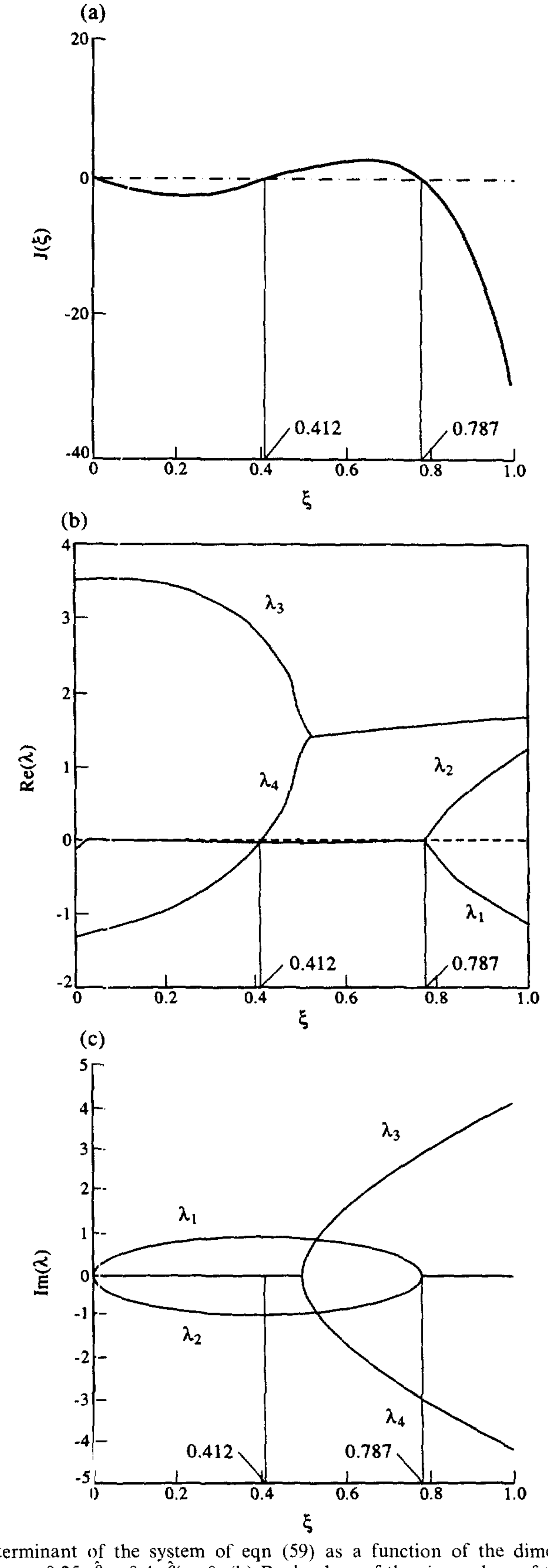

Fig. 6. (a) Determinant of the system of eqn (59) as a function of the dimensionless lateral compression for $\nu = 0.25$, $\hat{\beta} = 0.4$, $\hat{\beta}' = 0$. (b) Real values of the eigenvalues of the system of eqn (59) as a function of the dimensionless lateral compression for $\nu = 0.25$, $\hat{\beta} = 0.4$, $\hat{\beta}' = 0$. (c) Imaginary values of the eigenvalues of the system of eqn (59) as a function of the dimensionless lateral compression for $\nu = 0.25$, $\hat{\beta} = 0.4$, $\hat{\beta}' = 0$.

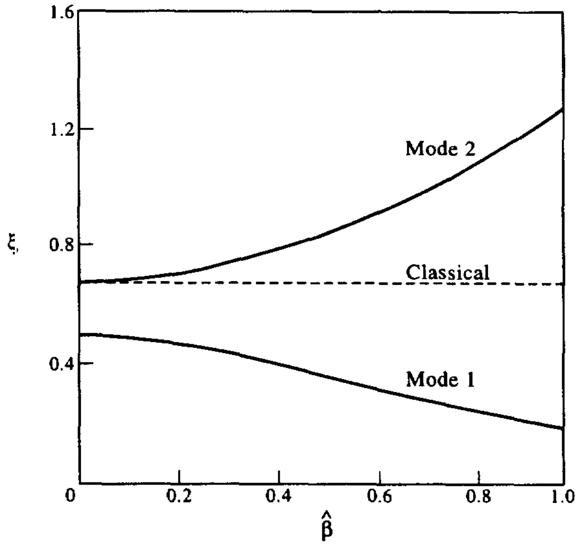

Fig. 7. Dimensionless buckling load of the elastic half-space as a function of the wave number $\hat{\beta}$ for Poisson's ratio $\nu = 0.25$ and surface energy parameter $\hat{\beta}' = 0$.

loading is assumed. It should be mentioned that the imaginary part of the eigenvalue associated with the minimum buckling load found above, is equal to zero at this buckling load. It can be shown that this mode (in this case first mode or mode-1) is characterized by a much faster attenuation and larger amplitude of the eigenstrain $\hat{\varepsilon}_{11}$ beneath the free surface as compared to the two higher buckling modes (Fig. 11). Figure 11 demonstrates that at the free surface, the absolute ratio of the normal eigenstrain ($\hat{\varepsilon}_{11}$) over the tangential eigenstrain ($\hat{\varepsilon}_{22}$) corresponding to the first buckling mode of the half-space with surface energy, is about 5.7, whereas the value of the same ratio $|\hat{\varepsilon}_{11}/\hat{\varepsilon}_{22}|$ corresponding to the second buckling mode is significantly lower ($\cong 0.2$). It is noted that the eigenstrain curves and the buckling load of the second mode are exactly the same with those corresponding to the first mode of the half-space without surface energy and characterized by the same values of $\hat{\beta} = 0.4$, $\nu = 0.25$.

Furthermore, Fig. 12 shows the dimensionless buckling stresses corresponding to the three buckling modes of the isotropic half-space versus the volumetric strain-gradient wave number $\hat{\beta}$ for various values of the material length ratio ℓ'/ℓ. For values of ℓ'/ℓ in the open-closed interval [0, 1) an extra buckling mode does not appear, the strain energy density w is positive definite (see Appendix A), and furthermore there is no appreciable effect of the surface energy parameter on the values of the buckling stresses corresponding to the two buckling modes. On the other hand, the diagram of Fig. 12 shows that the buckling stress for the first mode (or mode-1) is a decreasing function of $\hat{\beta}$, as is also happens for the second mode. However, the buckling stress corresponding to the first mode tends to the value of zero for a certain value of $\hat{\beta}$, depending on the relative surface energy parameter ℓ'/ℓ. Furthermore, a comparison between the results presented in Fig. 12 and the results presented in Fig. 7, shows that the second buckling mode in the presence of surface energy remains exactly the same as the first buckling mode of the half-space without surface energy. Hence, the mode-2 may be attributed to the effect of the volume energy strain-gradient parameter ℓ. The third buckling mode of the medium with surface energy is slightly different from the second mode of the medium without surface energy.

Of great interest is the derivation of a mode-1 surface instability criterion for the gradient-elastic half-space with surface energy. Figure 13 illustrates the relation of the relative surface energy parameter vs the volume energy wave number $\hat{\beta}$, which gives zero buckling stress for mode-1 and for Poisson's ratio $\nu = 1/4$. This relation has the form

$$\ell'/\ell = 1+0.1\hat{\beta}+0.32\hat{\beta}^2\,; \quad \nu = 1/4 \tag{64}$$

Also, in Fig. 13 the asymptotic curves

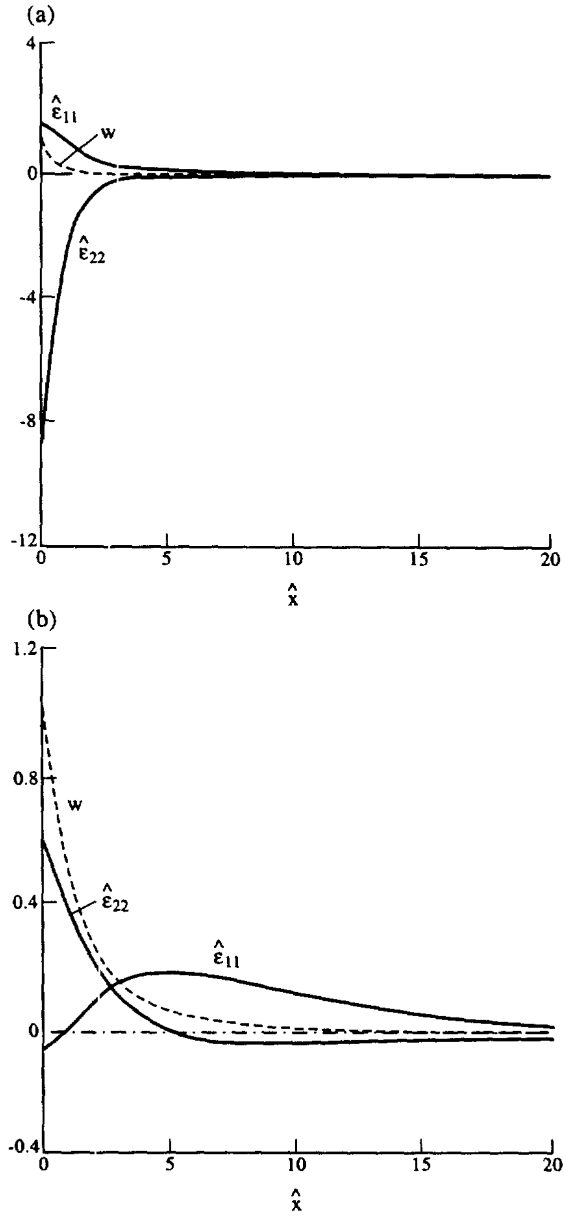

Fig. 8. Normalized depth x_1/ℓ vs the eigenstrains $\hat{\varepsilon}_{11}$, $\hat{\varepsilon}_{22}$ and eigenstrain energy density $w = \hat{w}(\hat{x}, 0)/\hat{w}(0, 0)$ for $\nu = 0.25$, $\hat{\beta} = 0.4$, $\hat{\beta}' = 0$; (a) First buckling mode, (b) Second buckling mode.

$$\ell'/\ell \cong 1+0.1\hat{\beta}, \quad \hat{\beta} \to 0;$$
$$\ell'/\ell \cong \sqrt{2}-0.76(1-\hat{\beta}), \quad \hat{\beta} \to 1 \tag{65}$$

have been plotted with dashed lines. From eqns (64) and (65) it can be seen that in the limit as $\hat{\beta} \to 0$ then $\ell'/\ell \to 1$ for a zero value of buckling stress. This result is also verified in the one kinematical degree-of-freedom half space problem presented in Appendix A. Note that the case of Appendix A does not take into account the influence of the volume energy strain-gradient parameter on the surface instability of the half-space. From the above it can also be inferred that in the limit as the material length ratio ℓ'/ℓ tends from the right to the value of one, the mode-1 buckling disappears. For pair of values $(\hat{\beta}, \ell'/\ell)$ falling below the curve of Fig. 13, or in other words, if the following stability condition is satisfied

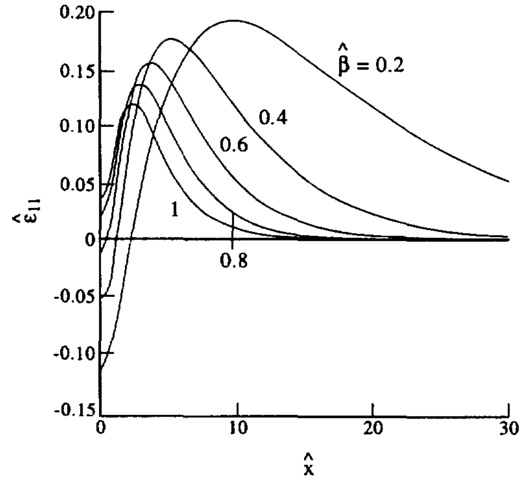

Fig. 9. Normalized depth $\hat{x} = x_1/\ell$ vs the eigenstrain $\hat{\varepsilon}_{11}$ associated with the second buckling mode plotted for five values of the wave number $\hat{\beta}(\nu = 0.25, \hat{\beta}' = 0)$.

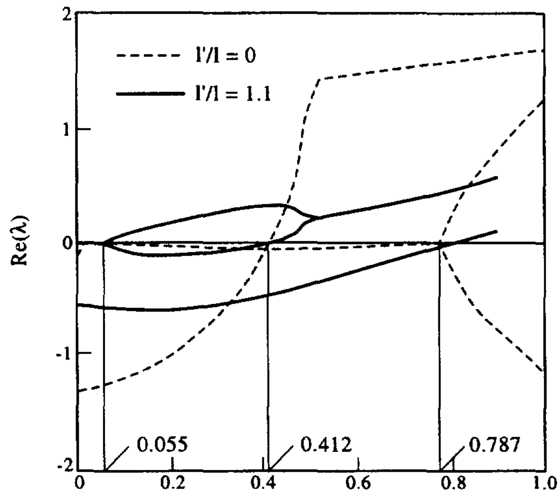

Fig. 10. Real values of eigenvalues of the system of eqn (59) as a function of the dimensionless lateral compression for two values of material length ratio ℓ'/ℓ ($\nu = 0.25, \hat{\beta} = 0.4$).

$$\ell'/\ell < 1+0.1\hat{\beta}+0.32\hat{\beta}^2\,; \quad \nu = 1/4 \tag{66}$$

the mode-1 buckling of the gradient-dependent elastic half-space with surface energy cannot take place. On the other hand, for the pairs falling above the characteristic curve eqn (64), or violate eqn (66), mode-1 surface instability that is characterized by a negative definite strain energy density w can occur at a certain stress level $\hat{\beta} \neq 0$. Inequality eqn (66) also implies the satisfaction of the "complementing" condition (Benallal *et al.*, 1988) for mode-1 instability

A further parametric study shows that for values of the relative surface energy parameter $\ell'/\ell > \sqrt{2}$, there exist only two non-trivial buckling modes, thus mode-1 disappears, with that corresponding to the volume energy term (mode-2 in Fig. 12) remaining always the same, and with the mode-3 (in Fig. 12) exhibiting first an increase and then an abrupt decrease with increasing $\hat{\beta}$. For even higher values of the surface energy parameter, mode-3 tends to the corresponding mode of the half-space without surface energy.

From the above analysis it may be remarked that stability is sometimes too severe a limitation to impose upon a system. There exist large classes of important practical problems where negative definite strain energy density can occur; for instance the surface instability

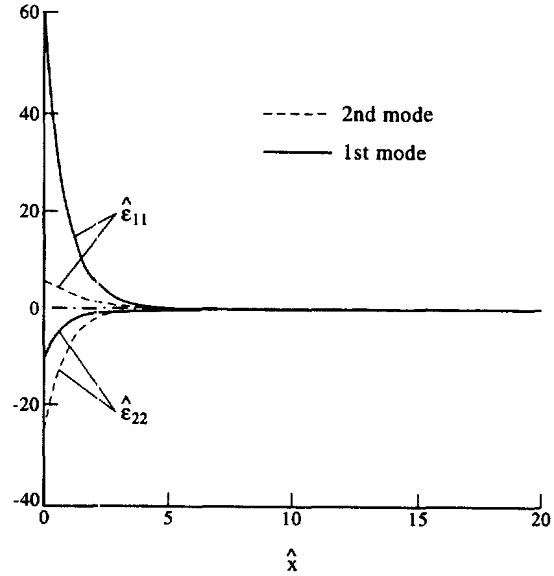

Fig. 11. Normalized depth x_1/ℓ vs the eigenstrains $\hat{\varepsilon}_{11}$, $\hat{\varepsilon}_{22}$ associated with the first and the second buckling mode ($\nu = 0.25$, $\hat{\beta} = 0.4$, $\ell'/\ell = 1.2$).

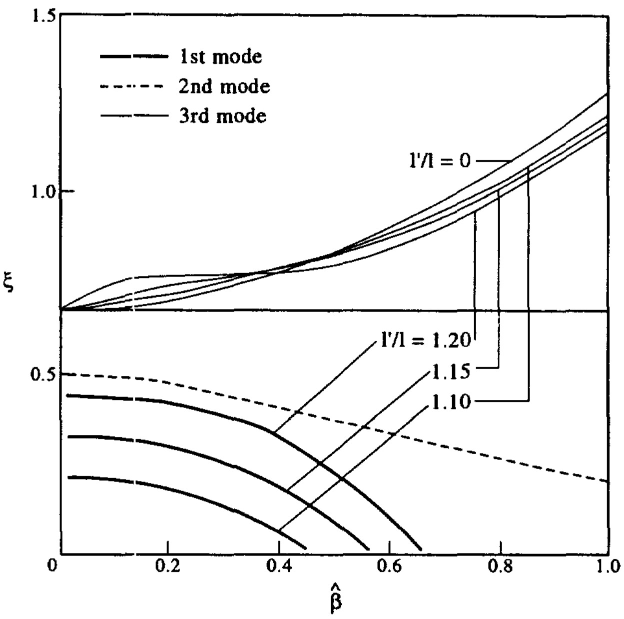

Fig. 12. Dimensionless buckling load of the elastic half-space as a function of the wave number $\hat{\beta}$ for Poisson's ratio $\nu = 0.25$ and for four values of the material length ratio ℓ'/ℓ.

problem at hand, which for $\ell'/\ell > 1$ gives realistic buckling strengths ($\sigma <<< G$), the bursting of balloons and of rock specimens at unconfined compression in the laboratory, or of rock pillars at great depths, the bowing and sudden exfoliation phenomena in marble and granite quarries when blocks and panels are cut, and many other.

Finally, the layer of thickness h with an array of surface-parallel Griffith cracks along the centre-line subjected to a large horizontal compression (Keer *et al.*, 1982) is approximated in the context of the present theory as a system of two layers (see Appendix B) of equal thickness $h/2$. The layers are characterized by the same elastic properties and volumetric strain-gradient parameter ℓ, zero surface energy parameter at the top and bottom surfaces $x_1 = 0$ and $x_1 = h$, and equal but with opposite sign surface energy parameter along their interface. The results obtained from this analysis for $\nu = 0.25$, $\hat{\beta} = 0.1$ and for three

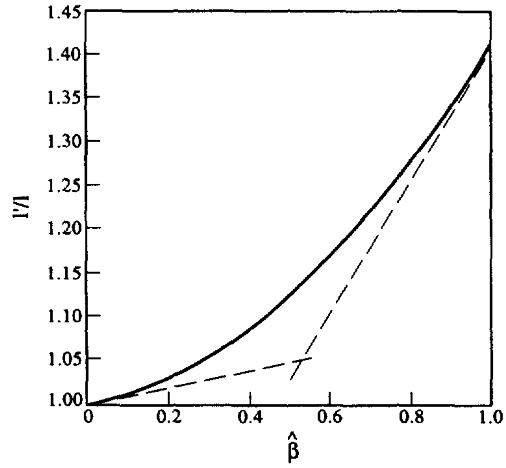

Fig. 13. Characteristic curve of relative surface energy parameter ℓ'/ℓ vs the volume energy wave number $\hat{\beta}$ which gives mode 1 surface instability for $\xi = 0$ (Poisson's ratio $\nu = 0.25$).

values of the material length ratio ℓ'/ℓ equal to 0, 0.8 and 1, respectively, are presented in Fig. 14. The comparison of this diagram with that presented by Keer *et al.* (1982) shows that the present higher-order continuum model can be used effectively for homogenization of elastic media with cracks. Also, from this comparison it can be inferred that the effect of $\ell' v_r$ with $v_r = \pm n_r$ can be interpreted as the homogenized effect of an array of surface-parallel, coplanar cracks. Results on the homogenization of elastic bodies containing periodic arrays of collinear cracks by using the present anisotropic gradient elasticity theory with surface energy, will be presented in a forthcoming publication. Hence, it would be advisable to look for the effect of the surface energy parameter by measuring the failure load of successively thinner single-layers of rock beams in folding (buckling) experiments (Handin *et al.*, 1972).

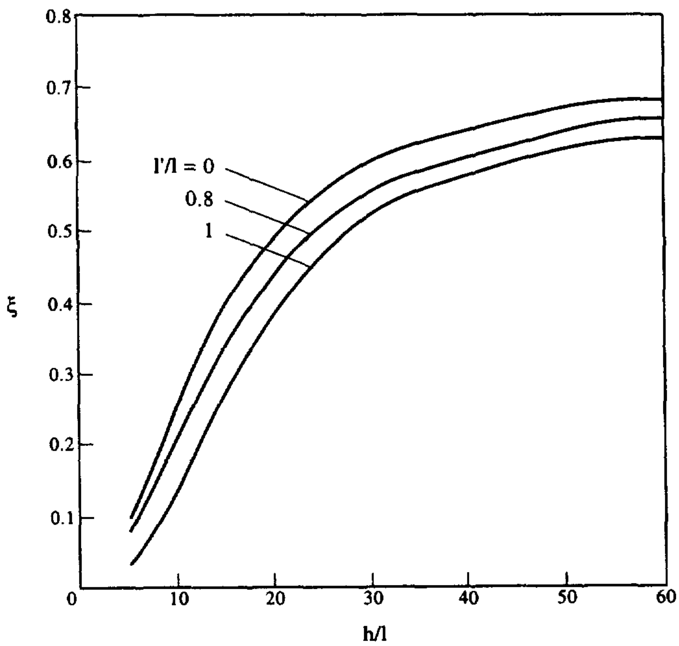

Fig. 14. Dimensionless buckling load of the layer with an interface along the centre line, as a function of the relative thickness h/ℓ of the layer, for three values of the material length ratio ℓ'/ℓ and for $\nu = 0.25$, $\hat{\beta}' = 0.1$.

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APPENDIX A: STRAIN AND SURFACE STRESS AT A FREE PLANE SURFACE OF A SOLID

Let the Cartesian coordinates be x_1, x_2, and x_3. Consider the deformation of an isotropic semi-infinite body $x_1 \geqslant 0$ due to a large uniform tensile stress $\sigma_{22} = \sigma$ ($\sigma > 0$), parallel with the free surface with outward unit normal vector $(\mathbf{n}_1\,\mathbf{n}_2\,\mathbf{n}_3) = (-1\,0\,0)$. Starting from a stress-free configuration, C_0, the body is stressed uniaxially, and C is the resultant configuration. The problem under consideration is formulated in terms of the first Piola–Kirchhoff stress π_{ij} with respect to current configuration, with $\Delta\pi_{ij}$ being its increment referred to the deformed initially stressed state C (see Section 3 of this paper). In the case of constant body forces, the equations of equilibrium for the incremental problem are given by

$$\partial_j \Delta\pi_{ij} = 0$$

For the traction-free surface of the half-space the following boundary conditions are valid

$$\Delta\pi_{11} = \Delta\pi_{21} = \mu_{111} = \mu_{112} = 0 \quad \text{on} \quad x_1 = 0, \quad -\infty < x_2 < \infty \tag{A1}$$

It is possible to assume, without loss of generality, the following displacement field

$$u_2 = u_2(x_1), u_1 = u_3 = 0 \tag{A2}$$

It can be shown that, in this problem, the quantities u_1, u_3 do not couple with u_2; these quantities satisfy homogeneous equations with homogeneous boundary conditions and therefore vanish identically. Upon substituting the strain-displacement relation into the stress–strain relations and the resulting expressions for the stresses into the stress–equation of equilibrium, we find only the following surviving displacement-equation of equilibrium

$$\left(1 - \frac{\ell^2}{(1+\xi)}\frac{\mathrm{d}^2}{\mathrm{d}x_1^2}\right)\frac{\mathrm{d}^2}{\mathrm{d}x_1^2}u_2 = 0 \tag{A3}$$

where $\xi = -\sigma_{22}/2G$.

The solution of (A3), vanishing at infinity, is

$$u_2(x_1) = c\exp\left(-\frac{\sqrt{1+\xi}}{\ell}x_1\right) \tag{A4}$$

with c to be an arbitrary real constant. The first three boundary conditions (A1) are satisfied identically, whereas the only remaining boundary condition along $x_1 = 0$ takes the form

$$\mu_{112} = 2G\left\{\ell'\frac{\mathrm{d}}{\mathrm{d}x_1} + \ell^2\frac{\mathrm{d}^2}{\mathrm{d}x_1^2}\right\}u_2 = 0 \quad \text{on} \quad x_1 = 0 \tag{A5}$$

which holds true for $v_r \equiv -n_r$ in definition equality (22). In turn, relationships (A5) and (A4) give the following homogeneous equation

$$c\left[-\frac{\ell'}{\ell}+\sqrt{1+\xi}\right]=0 \tag{A6}$$

From equation (A6) one may deduce that the only case which gives non-zero and exponentially decaying displacement with distance from the surface of the solid, that is $c \neq 0$, is the following

$$\frac{\ell'}{\ell}=\sqrt{1+\xi}\rightarrow\xi=\left(\frac{\ell'}{\ell}\right)^2-1 \tag{A7}$$

The above relation elucidates the importance of the surface strain gradient term ℓ' in determining surface effects. As it may be seen from (A7) and (A4) the presence of ℓ' gives rise to a skin effect or localized surface deformation, i.e. an exponential displacement and strain decay with distance from the free surface

$$u(x_1)=c\exp\left(-\frac{\ell'}{\ell}\frac{x_1}{\ell}\right)\rightarrow\varepsilon(x_1)=-\frac{\ell'}{\ell}\frac{c}{\ell}\exp\left(-\frac{\ell'}{\ell}\frac{x_1}{\ell}\right) \tag{A8}$$

wherein $u = u_2$, $\varepsilon = \mathrm{d}u_2/\mathrm{d}x_1$. It is known from measurements of low-energy electron-diffraction at nickel surfaces by Germer *et al.* (1961) that the displacement of the superficial layer of atoms toward the interior is five times as large as that of a next layer, demonstrating the very rapid decay from the free surface. Furthermore, eqn (A7) depicts that the effect of the volume and surface energy parameters is equivalent to the effect of initial stress. In the case of absence of the surface energy parameter ($\ell' = 0$) with $\ell \neq 0$, then from eqn (A7) it is found that $c \neq 0$ only if $\xi = -1$. As it can be seen from eqn (A4) this implies that $u = c$ and $\varepsilon = 0$. Hence, this case corresponds to a homogeneous state of deformation of the half-space and gives no surface phenomena. On the other hand, as it was shown above the surface energy parameter is associated with surface phenomena. The dependence of initial stress ξ on the relative surface energy parameter ℓ'/ℓ is shown in Fig. A1. From this figure is may be seen that if $\ell'/\ell = 0$ the half-space is under tension, with this tension to be the maximum. As ℓ'/ℓ increases from the value of zero, the initial tension of the medium decreases reaching the value of zero for $\ell'/\ell = 1$. At $\ell'/\ell = 1$ the initial stress changes sign and for $\ell'/\ell > 1$ becomes compressive in nature. That is, for values of the relative surface energy parameter higher than the value of one, the medium is under surface compression and it is no longer in a state of elastic equilibrium, or in other words, as it is shown below, its strain energy density function is negative definite.

The elastic strain energy density of the considered one-dimensional kinematical field is obtained from eqn (21) as follows

$$w = G\{\varepsilon^2+\ell^2\nabla\varepsilon\nabla\varepsilon+2\ell'\varepsilon\nabla\varepsilon\},\quad \nabla\equiv \mathrm{d}/\mathrm{d}x_1 \tag{A9}$$

Substituting in eqn (A9) the values for the strain and the strain-gradient by using (A8), we find

$$\hat{w}=\left\{1-\left(\frac{\ell'}{\ell}\right)^2\right\}\left(\frac{\ell'}{\ell}\right)^2\left(\frac{c}{\ell}\right)^2\exp\left(-2\frac{\ell'}{\ell}\frac{x_1}{\ell}\right),\quad \hat{w}=w/G \tag{A10}$$

Hence, in order for the strain energy density function to be positive definite the inequalities

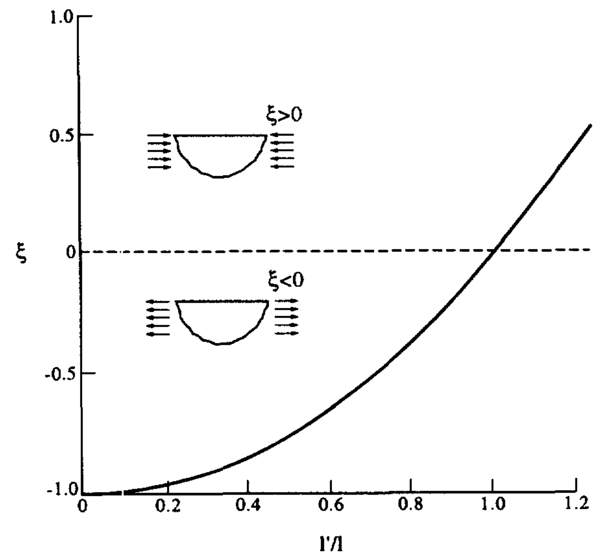

Fig. A1. Graphical representation of the relation of the dimensionless pre-stress ξ with the relative surface energy parameter ℓ'/ℓ.

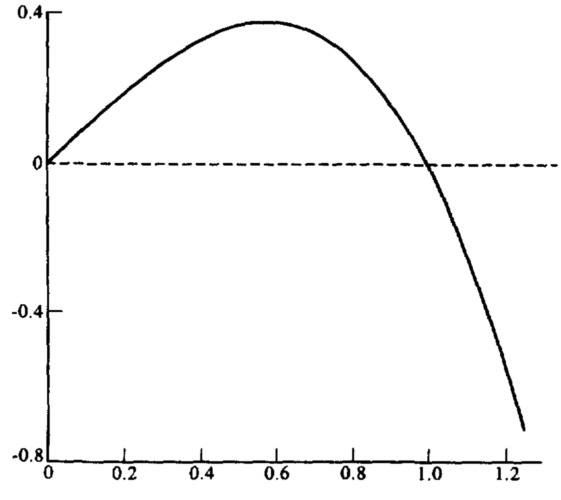

Fig. A2. Plot of normalized specific surface energy $\hat{\gamma} = \gamma/[(G/2)(c^2/\ell)]$ vs the relative surface energy parameter ℓ'/ℓ.

$$-1 < \frac{\ell'}{\ell} < 1, \quad \ell^2 > 0 \tag{A11}$$

must hold true. The condition $\ell'/\ell > -1$ is obtained if we put $v_r \equiv n_r$ instead of $v_r \equiv -n_r$ in definition equality eqn (22), whereas the second of eqn (A11) implies that ℓ is real.

Then we adopt the following definition for the average surface stress (or surface free energy)

$$\gamma = \int_V w \, \mathrm{d}V/A \tag{A12}$$

where A is the area of the free plane surface. After some manipulations we find the formula

$$\gamma = \frac{G}{2}\left\{1 - \left(\frac{\ell'}{\ell}\right)^2\right\}\frac{\ell' c^2}{\ell^2} \tag{A13}$$

This is also, for each surface, the energy per unit area required to separate the body along a plane, and $\gamma > 0$ if inequality (A11) holds true. Finally, the dependence of the normalized surface free energy on the relative surface energy parameter ℓ'/ℓ is displayed in Fig. A2.

APPENDIX B: ANALYSIS OF AN ISOLATED LAYER AND OF MULTILAYERED SYSTEMS

Let us consider the non-trivial, plane-strain deformation of a layer of thickness h, due to constant horizontal compression $\sigma_2 = -\sigma$, as shown in Fig. B1. The displacement amplitudes at the upper and lower boundaries of the layer are obtained directly from eqn (37)

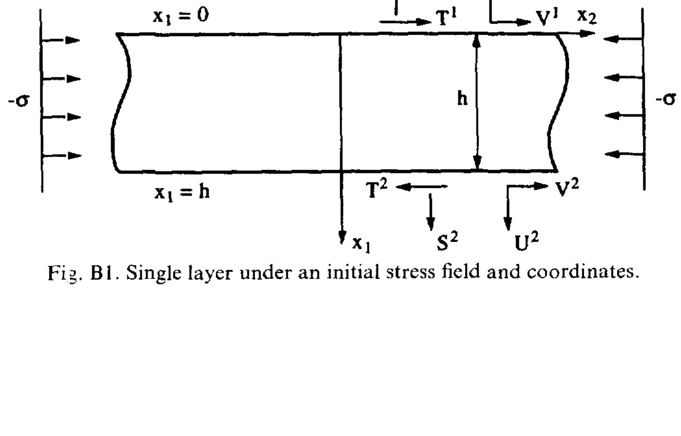

Fig. B1. Single layer under an initial stress field and coordinates.

$$U^1 = \sum_{i=1}^{8} A_i \hat{U}_i(0)\,; \quad V^1 = \sum_{i=1}^{8} K_i A_i \hat{V}_i(0)$$

$$U^2 = \sum_{i=1}^{8} A_i \hat{U}_i(\hat{h})\,; \quad V^2 = \sum_{i=1}^{8} K_i A_i \hat{V}_i(\hat{h}) \tag{B1}$$

where we have set $\hat{h} = h/\ell$ and the subscripts 1 and 2 denote the upper and lower faces of the layer, respectively. For example, in the case of eight pure imaginary roots $\pm\sqrt{z_i}$ $(i = 1, \ldots, 4)$ of the characteristic eqn (35) where z_i $(i = 1, \ldots, 4)$ are given by eqn (43), the displacement amplitude $U(x)$ is written as follows

$$U(x) = A_1 e^{-\beta \hat{Z}_1 x} + A_2 e^{-\beta \hat{Z}_2 x} + A_3 e^{-\beta \hat{Z}_3 x} + A_4 e^{-\beta \hat{Z}_4 x} + A_5 e^{-\beta \hat{Z}_5 x} + A_6 e^{-\beta \hat{Z}_6 x} + A_7 e^{-\beta \hat{Z}_7 x} + A_8 e^{-\beta \hat{Z}_8 x} \quad \hat{Z}_i \quad (i = 1, \ldots, 8) \in \Re^+ \tag{B2}$$

where $\hat{Z}_i = \hat{Z}_{4+i} = \mathrm{Im}(\sqrt{z_i})$, $(i = 1, \ldots, 4)$.

The normalized boundary tractions $s = \Delta\pi_{11}/G$, $t = \Delta\pi_{21}/G$ and double tractions $\mu_{11} = \mu_{111}/G$, $\mu_{12} = \mu_{112}/G$ are written through expressions (50), (53) and (32) as follows

$$s = \frac{\beta}{\ell} S \cos(\beta y)\,; \quad t = \frac{\beta}{\ell} \sin(\beta y)$$

$$\mu_{11} = M_{11} \cos(\beta y)\,; \quad \mu_{12} = M_{12} T \sin(\beta y) \tag{B3}$$

with

$$S = \sum_{i=1}^{8} A_i S_i\,; \quad S_i = c_{11} \hat{U}'_i/\beta + c_{12} K_i \hat{V}_i - 2g_* \hat{\beta}^2 (\hat{U}'''_i/\beta^3 - \hat{U}'_i/\beta)$$

$$T = \sum_{i=1}^{8} A_i T_i\,; \quad T_i = 2\{(1+\xi)(-U'_i + K_i \hat{V}'_i/\beta) - g_* \hat{\beta}^2 (\hat{U}_i - \hat{U}''_i/\beta^2 + K_i \hat{V}'''_i/\beta^3 - K_i \hat{V}'_i/\beta)\}$$

$$M_{11} = \sum_{i=1}^{8} A_i L_i\,; \quad L_i = 2g_* \{\hat{\beta}' \hat{U}'_i/\beta + \hat{\beta}^2 \hat{U}''_i/\beta\}$$

$$M_{12} = \sum_{i=1}^{8} A_i M_i\,; \quad M_i = 2g_* \{-\hat{\beta}'(-\hat{U}'_i + K_i \hat{V}'_i/\beta) + \hat{\beta}^2 (-\hat{U}'_i/\beta^2 + K_i \hat{V}''_i/\beta^2\} \tag{B4}$$

where $(\cdot)' \equiv \mathrm{d}/\mathrm{d}x$. The stress and double stress amplitudes S, T and M_{11}, M_{12}, respectively, at the upper and lower faces of the layer are written, in accordance with expressions (B1), as

$$S^1 = \sum_{i=1}^{8} A_i S_i(0)\,; \quad T^1 = \sum_{i=1}^{8} A_i T_i(0)$$

$$M^1_{11} = \sum_{i=1}^{8} A_i L_i(0)\,; \quad M^1_{12} = \sum_{i=1}^{8} A_i M_i(0)$$

$$S^2 = \sum_{i=1}^{8} A_i S_i(\hat{h})\,; \quad T^2 = \sum_{i=1}^{8} A_i T_i(\hat{h})$$

$$M^2_{11} = \sum_{i=1}^{8} A_i L_i(\hat{h})\,; \quad M^2_{12} = \sum_{i=1}^{8} A_i M_i(\hat{h}) \tag{B5}$$

In matrix form the resulting homogeneous system of equations for the case of zero tractions and double tractions amplitudes at the upper $(i = 1)$ and lower $(i = 2)$ surfaces of the layer is written

$$[Y]\{A\} = \{0\}$$

or, alternatively

$$\begin{bmatrix} S_1(0) & S_2(0) & S_3(0) & S_4(0) & S_5(0) & S_6(0) & S_7(0) & S_8(0) \\ T_1(0) & T_2(0) & T_3(0) & T_4(0) & T_5(0) & T_6(0) & T_7(0) & T_8(0) \\ L_1(0) & L_2(0) & L_3(0) & L_4(0) & L_5(0) & L_6(0) & L_7(0) & L_8(0) \\ M_1(0) & M_2(0) & M_3(0) & M_4(0) & M_5(0) & M_6(0) & M_7(0) & M_8(0) \\ S_1(\hat{h}) & S_2(\hat{h}) & S_3(\hat{h}) & S_4(\hat{h}) & S_5(\hat{h}) & S_6(\hat{h}) & S_7(\hat{h}) & S_8(\hat{h}) \\ T_1(\hat{h}) & T_2(\hat{h}) & T_3(\hat{h}) & T_4(\hat{h}) & T_5(\hat{h}) & T_6(\hat{h}) & T_7(\hat{h}) & T_8(\hat{h}) \\ L_1(\hat{h}) & L_2(\hat{h}) & L_3(\hat{h}) & L_4(\hat{h}) & L_5(\hat{h}) & L_6(\hat{h}) & L_7(\hat{h}) & L_8(\hat{h}) \\ M_1(\hat{h}) & M_2(\hat{h}) & M_3(\hat{h}) & M_4(\hat{h}) & M_5(\hat{h}) & M_6(\hat{h}) & M_7(\hat{h}) & M_8(\hat{h}) \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \\ A_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{B6}$$

The above homogeneous system of eqn (B6) has non-trivial solutions in terms of the integration constants involved only if the determinant of the system matrix is singular, i.e. eqn (60) is valid.

The previous results make it possible to obtain very simply and in a systematic way the stability equations for a system of n superposed adhering layers under initial stress. However, in order to solve this problem we must first consider, in the context of anisotropic gradient elasticity theory with surface energy, a class of interface problems where two different gradient-dependent elastic materials are bonded by a plane (say $x_1 = 0$) and their volumetric coefficients are the same ($\ell_1 = \ell_2 = \ell$). In this case, we can establish the following continuity requirements

$$[u_i] = 0 \tag{B7}$$

$$[\partial_1 u_i] = 0 \tag{B8}$$

$$[\partial_1^3 u_i] = 0 \tag{B9}$$

$$[\mu_{11i}] = 0 \tag{B10}$$

$$[\Delta\pi_{i1}] \equiv 0 \quad \text{(identity)} \tag{B11}$$

where $i = 1, 2$, $[\cdot]$ denotes the jump of the quantity across a discontinuity line, and last condition is identically satisfied if eqns (B8) and (B9) are met. This identity can be verified by recourse to Maxwell's theorem which states that if a function is continuous across a geometrical discontinuity, say $[u_i] = 0$, then only the normal derivative of this function may be discontinuous across the discontinuity, i.e. $[\partial_2 u_i] = 0$ (Truesdell and Toupin, 1960).

Next, following Papamichos *et al.* (1990) and Vardoulakis and Sulem (1995) a global $(\bar{x}_1, \bar{x}_2)$ coordinate system is introduced, as shown in Fig. B2. The layers are numbered from $j = 1$ to n starting at the top. By assuming perfect adherence at the interfaces, then eqns (B7), (B8), (B10) and (B11) must be valid along the interfaces $i = 1$ to $n+1$. Under these conditions the equations for the buckling of the system of layers are derived immediately from the results obtained for the single layer, provided that all local coordinates are expressed in the global coordinate system. In accordance with expression (B6), the amplitudes of the incremental stresses, double stresses, displacements, and displacement gradients for the ith interface of the jth layer can be assembled in matrix form as follows

$$\begin{bmatrix} U^{ij} \\ V^{ij} \\ Y^{ij} \\ W^{ij} \\ S^{ij} \\ T^{ij} \\ L^{ij} \\ M^{ij} \end{bmatrix} = \begin{bmatrix} \hat{U}_1^{ij} & \hat{U}_2^{ij} & \hat{U}_3^{ij} & \hat{U}_4^{ij} & \hat{U}_5^{ij} & \hat{U}_6^{ij} & \hat{U}_7^{ij} & \hat{U}_8^{ij} \\ \hat{V}_1^{ij} & \hat{V}_2^{ij} & \hat{V}_3^{ij} & \hat{V}_4^{ij} & \hat{V}_5^{ij} & \hat{V}_6^{ij} & \hat{V}_7^{ij} & \hat{V}_8^{ij} \\ Y_1^{ij} & Y_2^{ij} & Y_3^{ij} & Y_4^{ij} & Y_5^{ij} & Y_6^{ij} & Y_7^{ij} & Y_8^{ij} \\ W_1^{ij} & W_2^{ij} & W_3^{ij} & W_4^{ij} & W_5^{ij} & W_6^{ij} & W_7^{ij} & W_8^{ij} \\ S_1^{ij} & S_2^{ij} & S_3^{ij} & S_4^{ij} & S_5^{ij} & S_6^{ij} & S_7^{ij} & S_8^{ij} \\ T_1^{ij} & T_2^{ij} & T_3^{ij} & T_4^{ij} & T_5^{ij} & T_6^{ij} & T_7^{ij} & T_8^{ij} \\ L_1^{ij} & L_2^{ij} & L_3^{ij} & L_4^{ij} & L_5^{ij} & L_6^{ij} & L_7^{ij} & L_8^{ij} \\ M_1^{ij} & M_2^{ij} & M_3^{ij} & M_4^{ij} & M_5^{ij} & M_6^{ij} & M_7^{ij} & M_8^{ij} \end{bmatrix} \begin{bmatrix} A_1^j \\ A_2^j \\ A_3^j \\ A_4^j \\ A_5^j \\ A_6^j \\ A_7^j \\ A_8^j \end{bmatrix}$$

or

$$\{X^{ij}\} = [F^{ij}]\{A^j\} \tag{B12}$$

where Y^{ij} and W^{ij} denote the amplitudes of the displacement gradients $\partial_1 u_1$, $\partial_1 u_2$, respectively, along the ith interface corresponding to the jth layer. By requiring continuity of the incremental displacements, displacement gradients, tractions and double tractions at all interfaces, the integration constants of every layer are linked to the integration constants of the top layer as follows

$$\{A^n\} = [F]\{A^1\}; \quad [F] = [F^n]\ldots[F^2], \quad [F^k] = [F^{k,k}]^{-1}[F^{k,k-1}] \tag{B13}$$

In order to formulate the eigenvalue problem we have to consider boundary conditions only at the upper and lower boundary surfaces of the layered medium. As an example, the case of zero tractions and double tractions

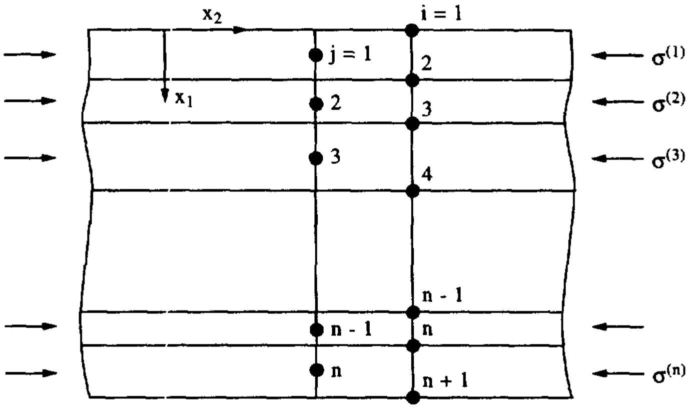

Fig. B2. Periodic laminated medium under lateral compression. The number of layers is infinite.

at the upper boundary surface ($i = 1, j = 1$) and at the lower boundary surface ($i = n+1, j = n$), is considered. These boundary conditions can be written in matrix form as

$$[Y^1]\{A^1\} = \{0\}$$
$$[Y^n]\{A^n\} = \{0\} \tag{B14}$$

By taking into account expression (B13), the matrix eqn (B14) can be assembled into a homogeneous algebraic system of equations for the integration constants $\{A^1\}$

$$\begin{bmatrix} [Y^2][F] \\ [Y^1] \end{bmatrix}\{A^1\} = \{0\} \quad \text{or} \quad [Y]\{A^1\} = \{0\} \tag{B15}$$

The resulting homogeneous system of equations has non-trivial solutions in terms of the integration constants involved if the determinant $[Y]$ is singular, that is eqn (60) holds true. This provides an equation whose roots give the corresponding eigenvalues.